Discrete Mathematics

Sets and set operations

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Basic discrete structures

- **Discrete math =**
  - study of the discrete structures used to represent discrete objects

- Many discrete structures are built using **sets**
  - **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    $V = \{ a, e, i, o, u \}$
  - **First seven prime numbers**.

    $X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

Representing sets

Representing a set by:

1) Listing (enumerating) the members of the set.

2) Definition by property, using the set builder notation

{x| x has property P}.

Example:

- Even integers between 50 and 63.

1) E = {50, 52, 54, 56, 58, 60, 62}

2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**

Important sets in discrete math

- **Natural numbers:**
  - $\mathbf{N} = \{0,1,2,3, \ldots\}$

- **Integers**
  - $\mathbf{Z} = \{\ldots, -2,-1,0,1,2, \ldots\}$

- **Positive integers**
  - $\mathbf{Z}^+ = \{1,2, 3\ldots.\}$

- **Rational numbers**
  - $\mathbf{Q} = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$

- **Real numbers**
  - $\mathbf{R}$

Russell's paradox

Cantor's naive definition of sets leads to Russell's paradox:

- **Let $S = \{ x \mid x \notin x \}$,**

  is a set of sets that are not members of themselves.
- **Question: Where does the set S belong to?**
  - Is $S \in S$ or $S \notin S$?
- **Cases**
  - **$S \in S$?:** S does not satisfy the condition so it must hold that **$S \notin S$ (or $S \in S$ does not hold)**
  - **$S \notin S$?:** S is included in the set S and hence $S \notin S$ does not hold
- **A paradox: we cannot decide if S belongs to S or not**
- **Russell's answer: theory of types – used for sets of sets**

Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- **$\{1,2,3\} = \{3,1,2\} = \{1,2,1,3,2\}$**

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

Example: Are $\{1,2,3,4\}$ and $\{1,2,2,4\}$ equal?

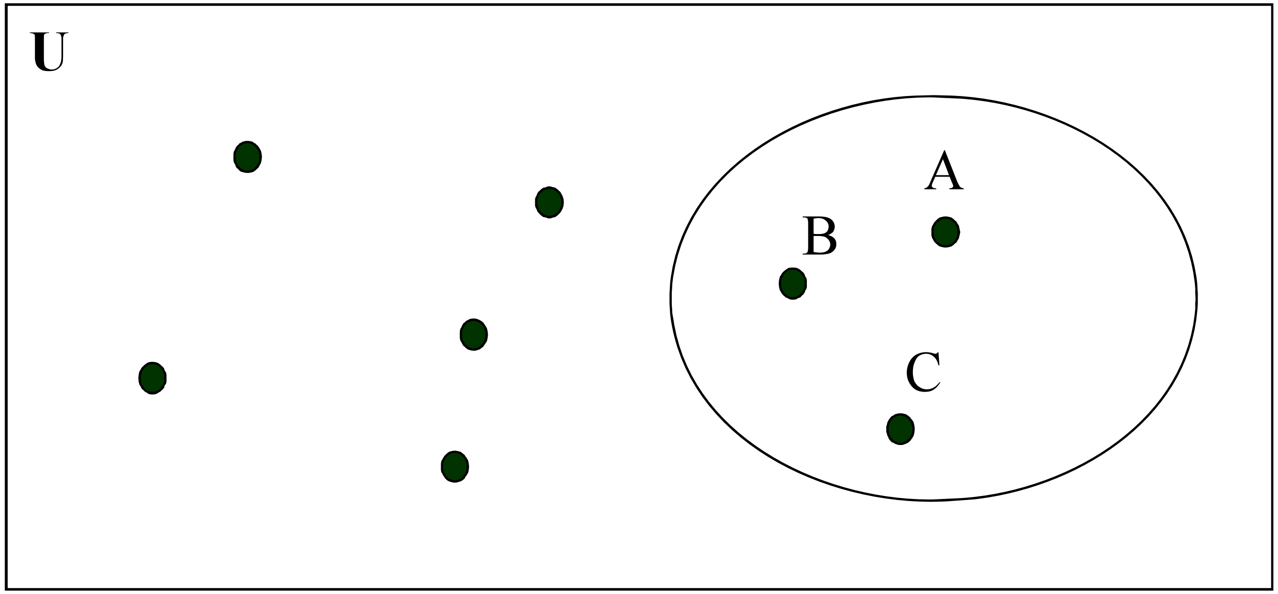
No!

Special sets

- **Special sets:**
  - **The <u>universal set</u>** is denoted by **U:** the set of all objects under the consideration.
  - **The <u>empty set</u>** is denoted as $\varnothing$ **or { }.**

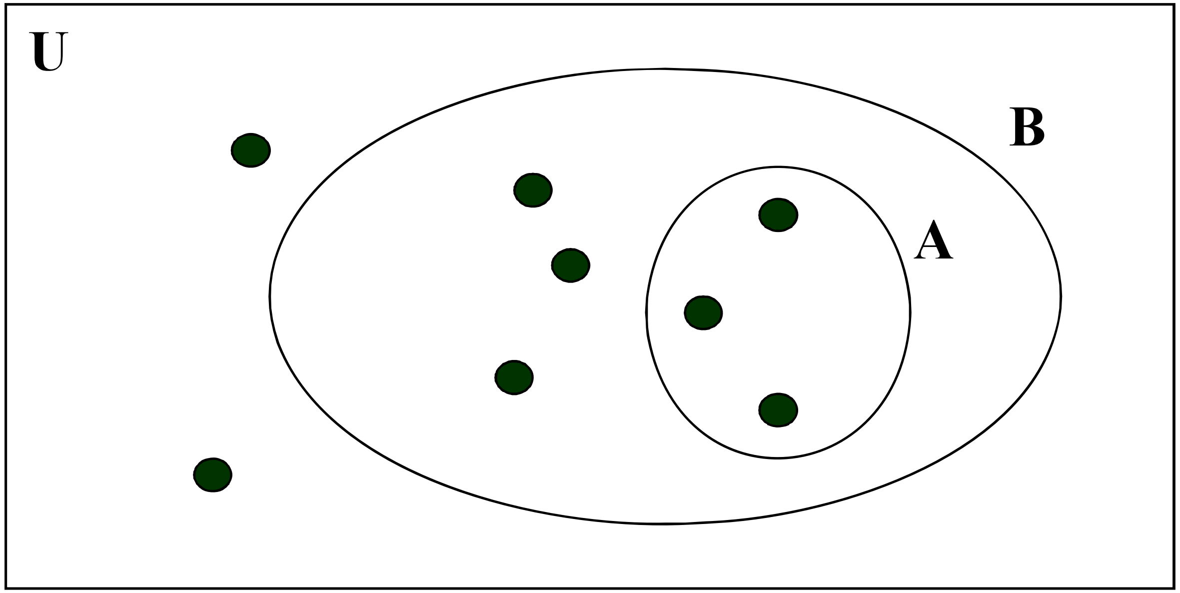
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **A $\subseteq$ B** to indicate **A is a subset of B**.



- Alternate way to define A is a subset of B:

  $\forall x\ (x \in A) \rightarrow (x \in B)$

Empty set/Subset properties

Theorem $\varnothing \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x \, (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S

  $\forall x \, (x \in \varnothing \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \varnothing$ is **always False**
- Then the implication is **always True.**

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

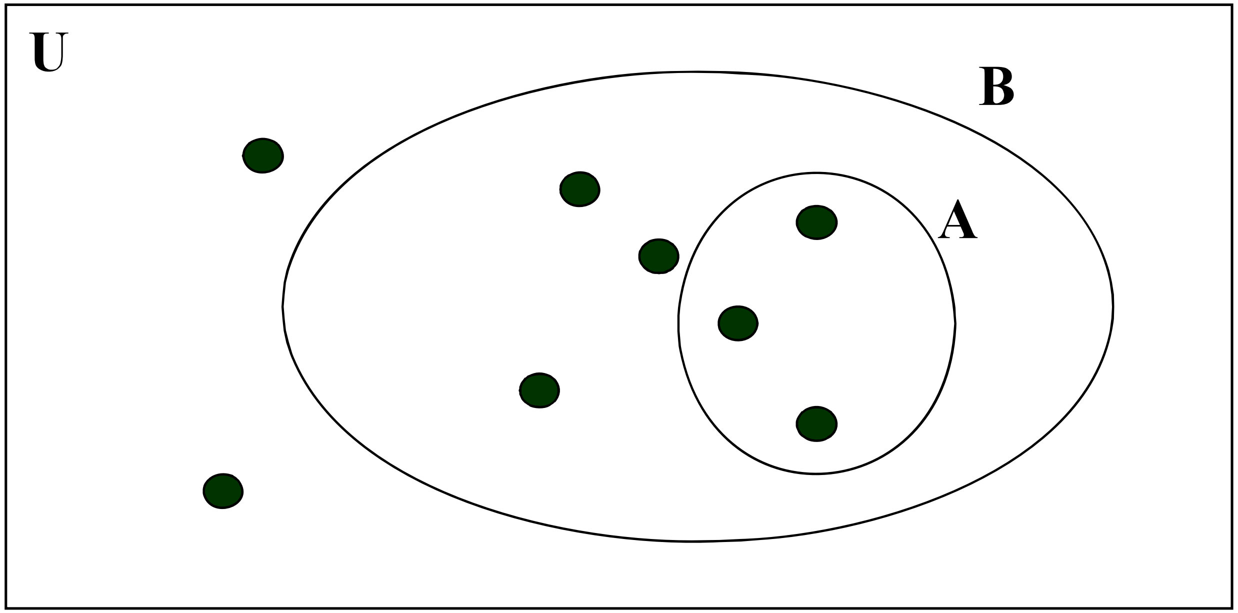
- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.

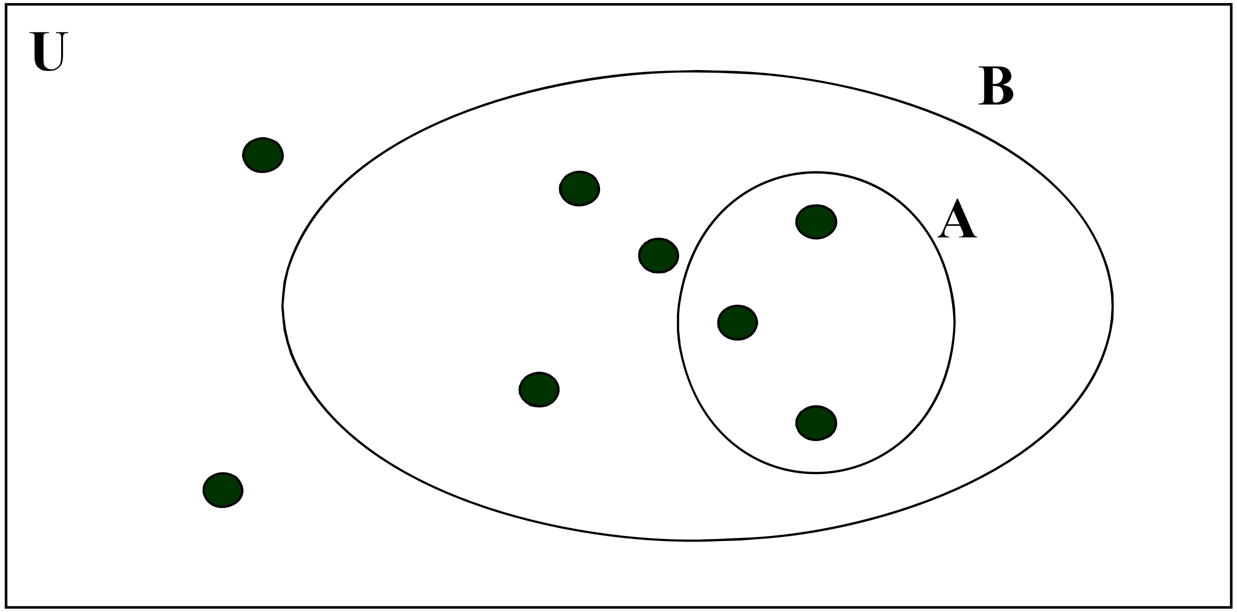
A proper subset

Definition: A **set A** is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



A proper subset

Definition: A **set A** is said to be a **proper subset** of B if and only if $\mathbf{A \subseteq B}$ and $\mathbf{A \neq B}$. We denote that A is a proper subset of B with the notation $A \subset B$.



Example: $A=\{1,2,3\}$ $B =\{1,2,3,4,5\}$

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |**.

Examples:

- **V={1 2 3 4 5}**

  | V | = 5

- **A={1,2,3,4, …, 20}**

  |A| =20

- **| ∅ | = 0**

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }

- The set of reals is an infinite set.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = \{ \emptyset \}$
- What is the cardinality of $P(\emptyset)$? $| P(\emptyset) | = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \emptyset, \{1\} \}$
- $|P(\{1\})| = 2$

Power set

- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\} | = 8$

- **If S is a set with $|S| = n$ then $| P(S) | = ?$**

Power set

- $P(\ \{1\}\ ) = \{\ \varnothing, \{1\}\ \}$
- $|P(\{1\})| = 2$

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- $|P(\{1,2,3\}\ | = 8$

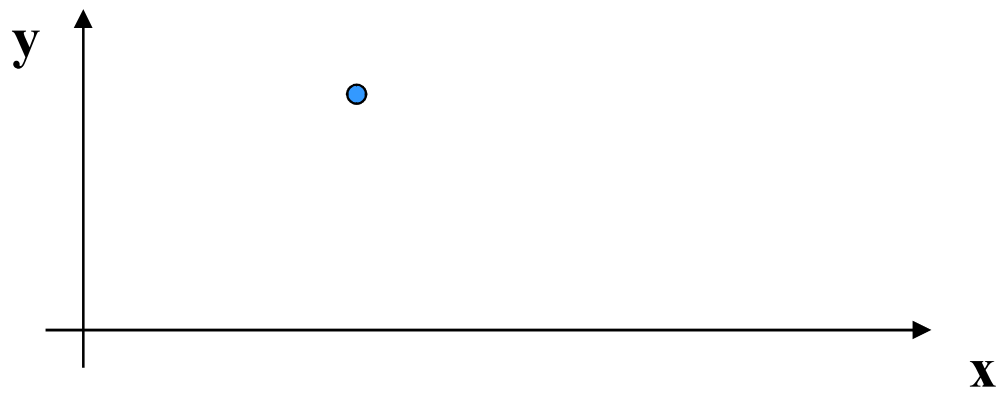
- **If S is a set with $|S| = n$ then $|\ P(S)\ | = 2^n$**

N-tuple

- Sets are used to represent unordered collections.
- **Ordered-n tuples** are used to represent an ordered collection.

Definition: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, N ≥ 2.

Example:



- Coordinates of a point in the 2-D plane (12, 16)

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$.

Examples:

- $S = \{1,2\}$ and $T = \{a,b,c\}$
- $S \times T = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$
- $T \times S = \{ (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) \}$
- Note: $S \times T \neq T \times S$!!!!

Cardinality of the Cartesian product

- $|S \times T| = |S| * |T|$.

Example:

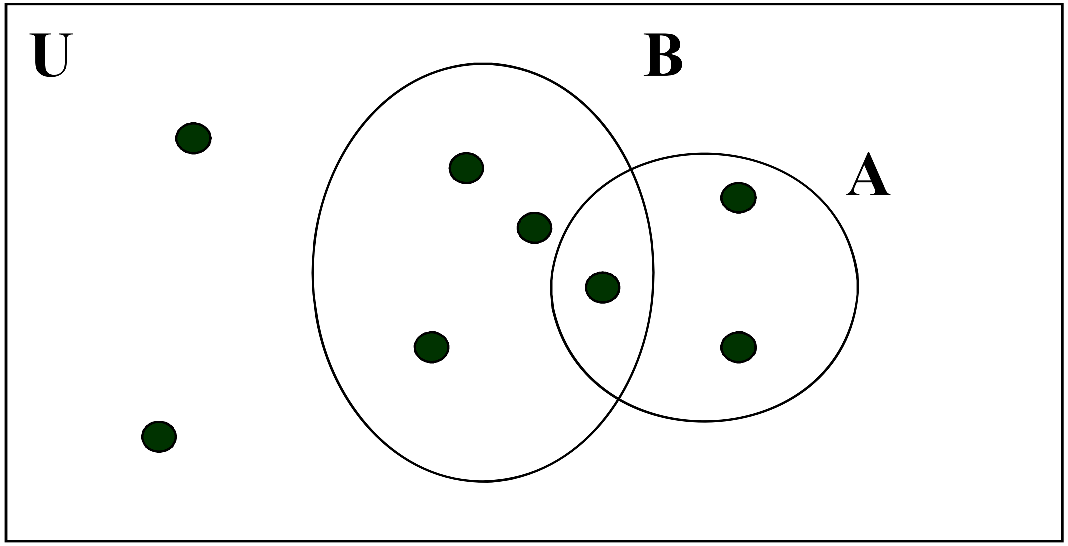
- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- $|A \times B| = 9$
- $|A|=3$, $|B|=3$ → $|A|\ |B|= 9$

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Set operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.

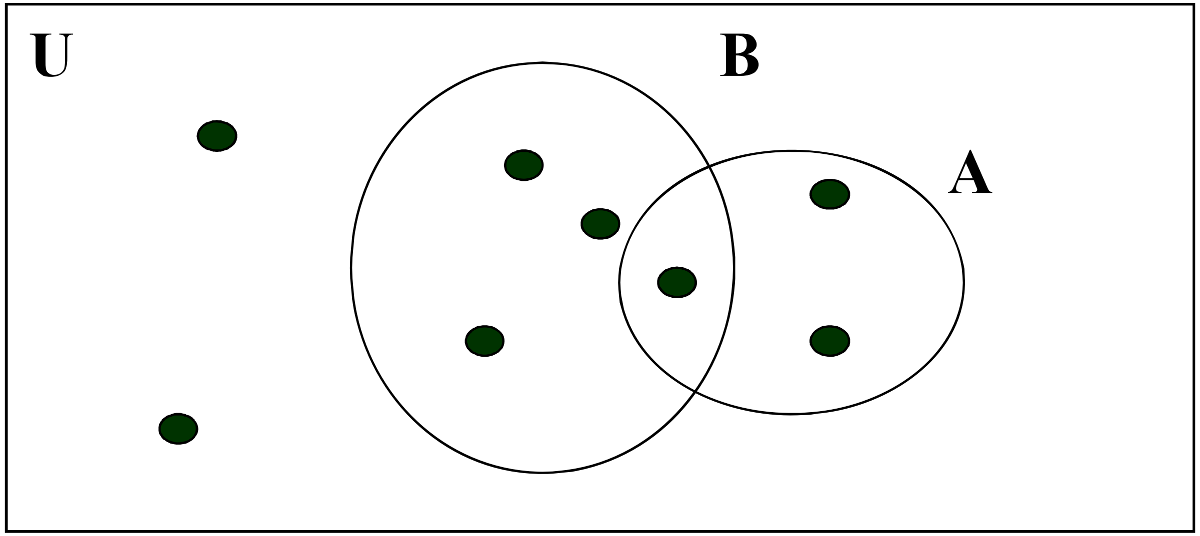


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



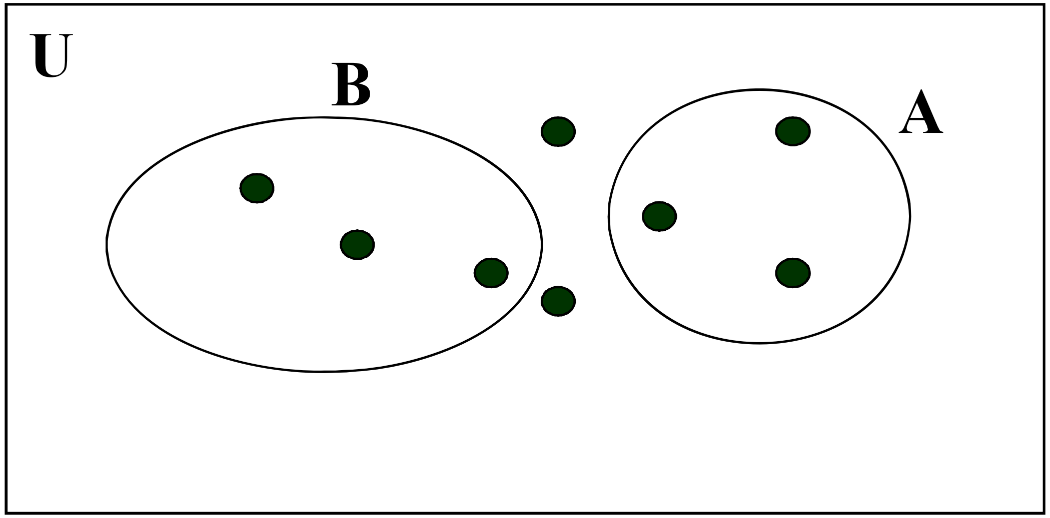
Example:

- $A = \{1,2,3,6\}$ $B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \varnothing$.



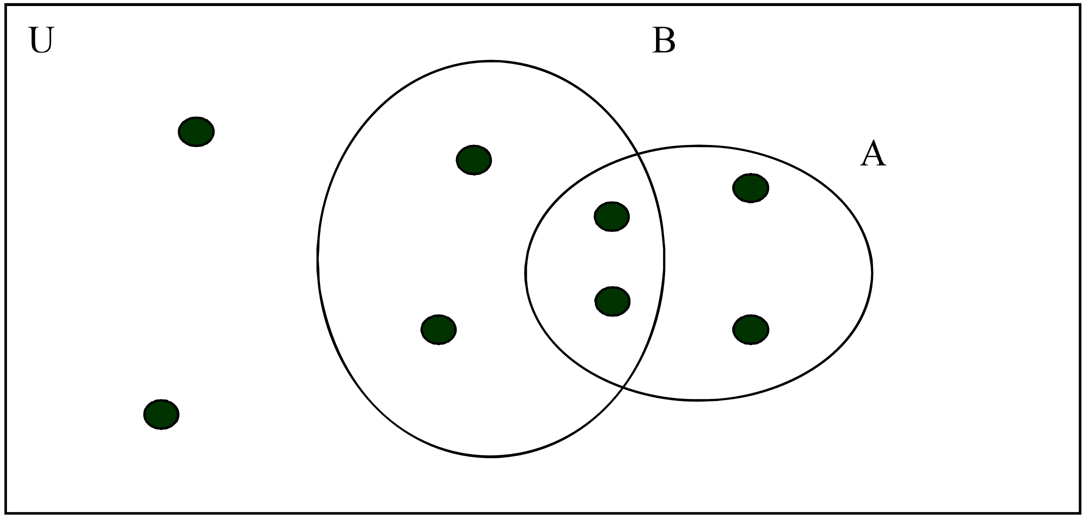
Example:

- $A=\{1,2,3,6\}$ $B=\{4,7,8\}$ Are these disjoint?
- Yes.
- $A \cap B = \varnothing$

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

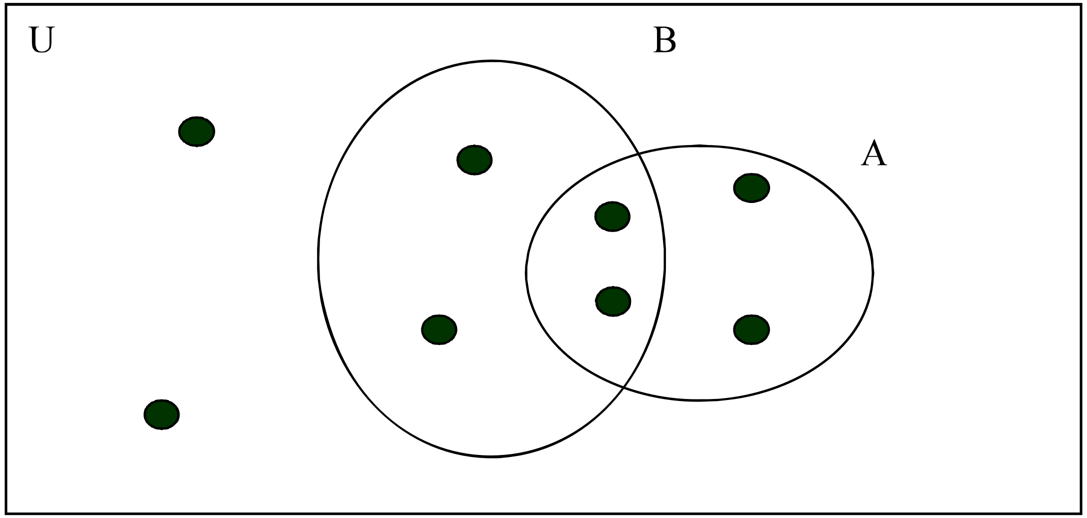


- Why this formula?

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

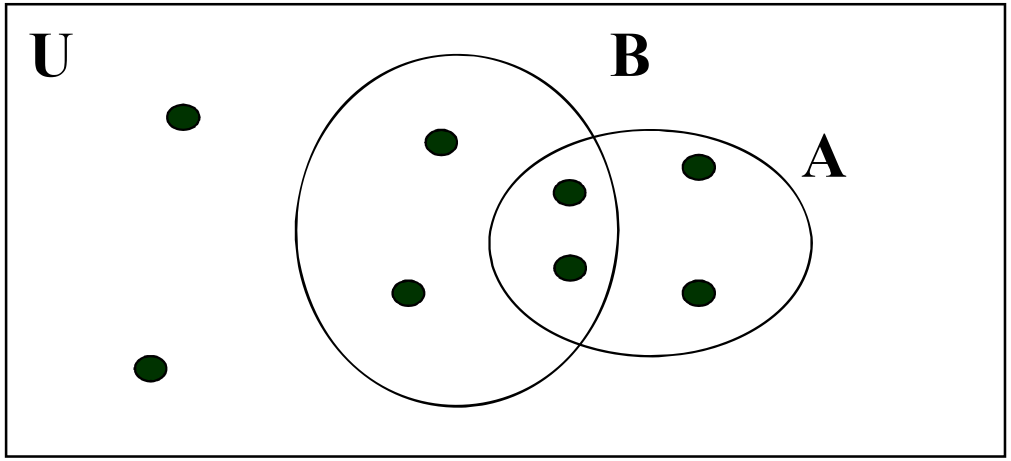


- Why this formula? Correct for an over-count.
- More general rule:
  - **The principle of inclusion and exclusion.**

Set difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



Example: A= {1,2,3,5,7} B = {1,5,6,8}

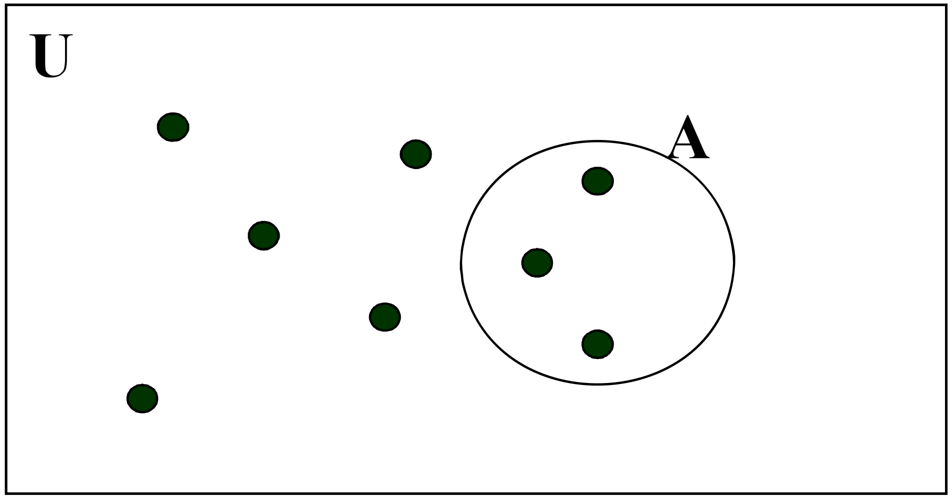
- A - B ={2,3,7}

Complement of a set

Definition: Let U be the **universal set**: the set of all objects under the consideration.

Definition: The **complement of the set A**, denoted by $\bar{A}$, is the complement of A with respect to U.

- Alternate: $\bar{A} = \{ x \mid x \notin A \}$



Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A} = \{2,4,6,8\}$

Set identities

Set Identities (analogous to logical equivalences)

- **Identity**
  - $A \cup \varnothing = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \varnothing = \varnothing$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \varnothing$

Set identities

- **Set identities can be proved using membership tables.**
- List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.
- Prove: $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Generalized unions and itersections

Definition: The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ $i = 1,2,\ldots,n$
- $$\bigcup_{i=1}^{n} A_i = \{1,2,\ldots,n\}$$

Generalized unions and intersections

Definition: The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \ldots \cap A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n

$$\bigcap_{i=1}^{n} A_i = \{ 1 \}$$

Computer representation of sets

- **How to represent sets in the computer?**
- **One solution: Data structures like a list**
- **A better solution:**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}
  - Computer representation: B = 10001

Computer representation of sets

Example:

- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\overline{A} = 10110$