Discrete Mathematics

Functions II

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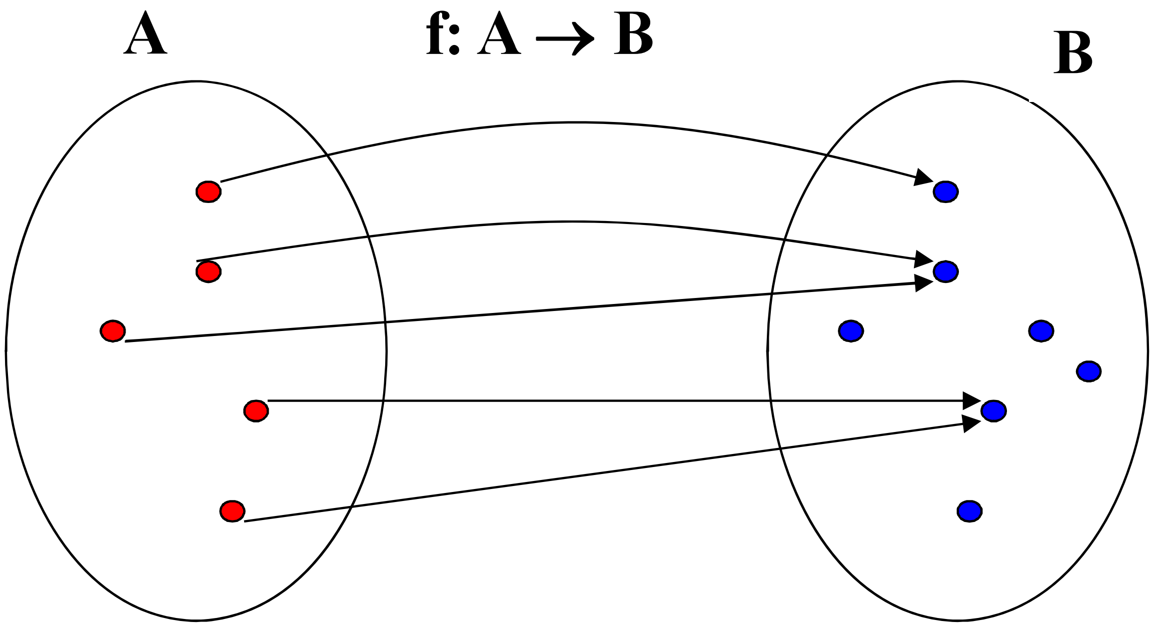
Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted $f : A \rightarrow B$, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



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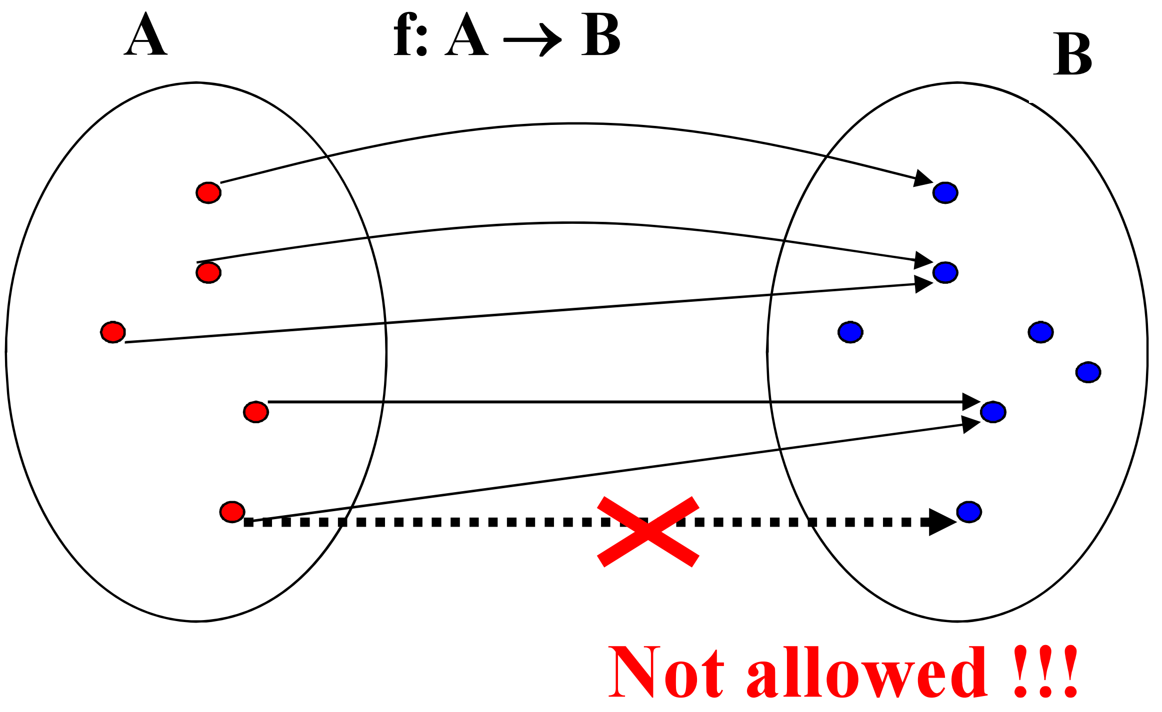
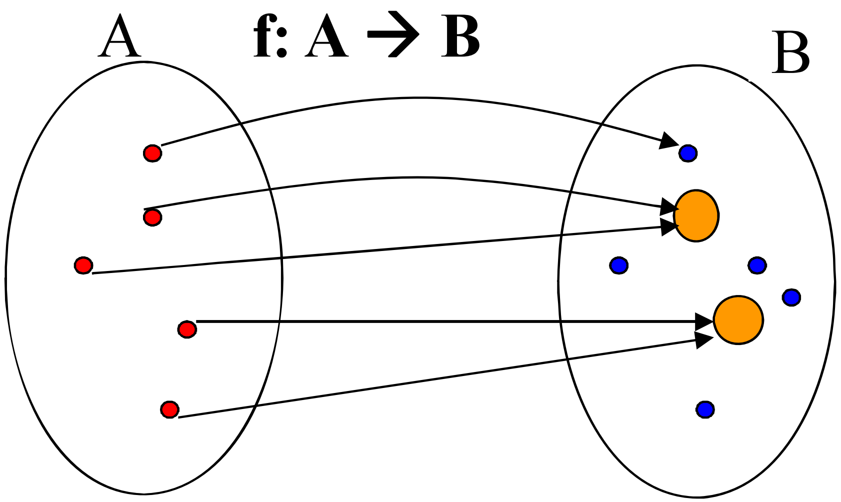


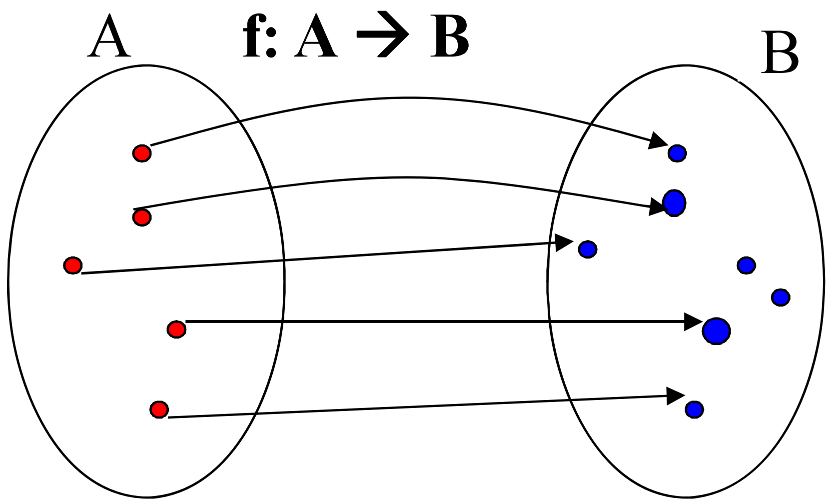
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternative: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective function

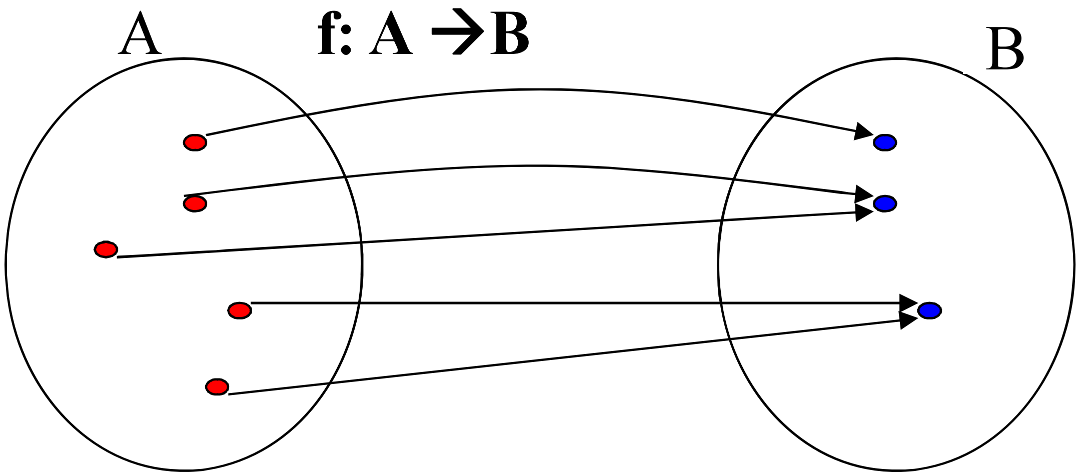


Injective function

Surjective function

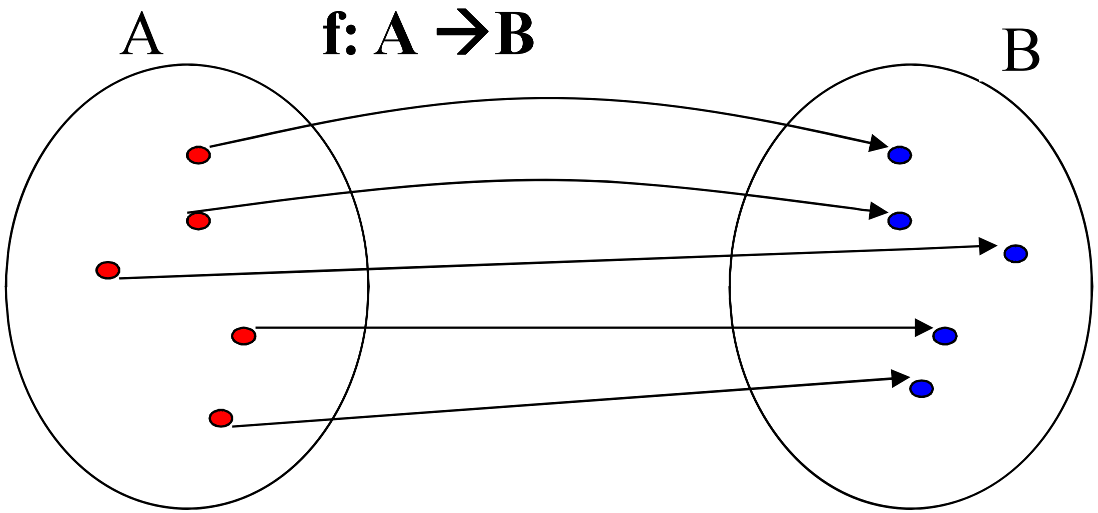
Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one (injection) and onto (surjection)**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **?**

Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

Bijective functions

Example 2:

- Define $g : W \to W$ (whole numbers), where $g(n) = \lfloor n/2 \rfloor$ (floor function).
    - $0 \to \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
    - $1 \to \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
    - $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
    - $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- $\ldots$
- Is g a bijection?

Bijective functions

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  - $0 \to \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
  - $1 \to \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
  - $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- …
- Is g a bijection?
  - **No.** g is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.

Bijective functions

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

→ **A is finite** and **f is one-to-one (injective)**

- Is f an **onto function (surjection)**?

Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?

Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
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← **A is finite and f is an onto function**

- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Bijective functions

Theorem. Let f be a function from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- **Example:**
  - $f : Z \to Z$, where $f(z) = 2 * z$.
  - f is one-to-one but not onto.
    - $1 \to \mathbf{2}$
    - $2 \to \mathbf{4}$
    - $3 \to \mathbf{6}$
  - 3 has no pre-image.

Functions on real numbers

Definition: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- $(f1 + f2)(x) = f1(x) + f2(x)$
- $(f1 * f2)(x) = f1(x) * f2(x)$.

Examples:

- **Assume**
  - $f1(x) = x - 1$
  - $f2(x) = x^3 + 1$

then

- $(f1 + f2)(x) = x^3 + x$
- $(f1 * f2)(x) = x^4 - x^3 + x - 1$.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Example:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?

Increasing and decreasing functions

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Example:

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?
- **Proof .**

  For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

  Thus g is strictly increasing.

Increasing and decreasing functions

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Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

Increasing and decreasing functions

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Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A: A \to A$ where $i_A(x) = x$.

Example:

- Let $A = \{1,2,3\}$

Then:

- $i_A(1) = ?$

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A: A \to A$ where $i_A(x) = x$.

Example:

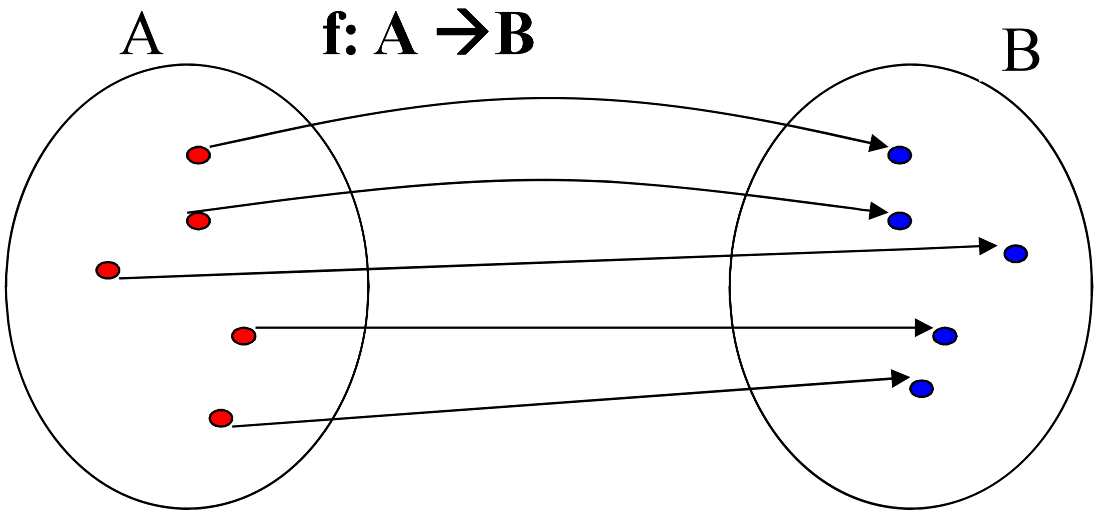
- Let $A = \{1,2,3\}$

Then:

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

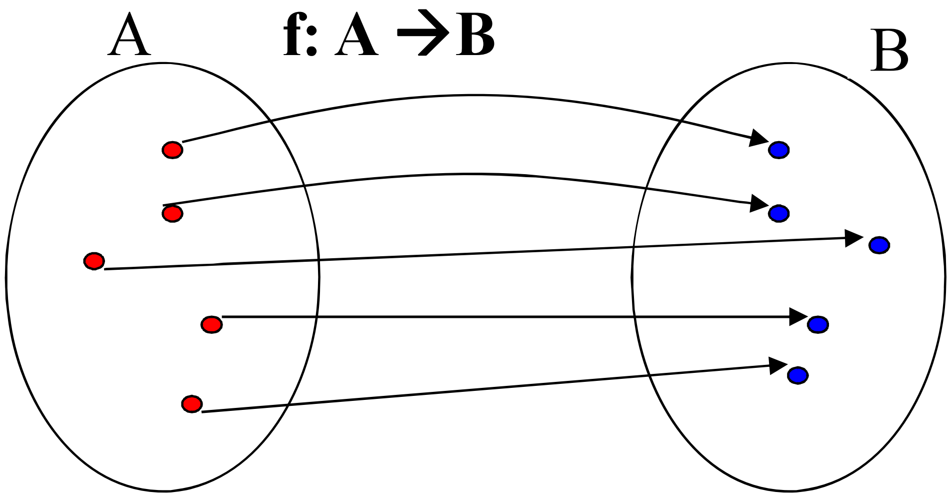
Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.

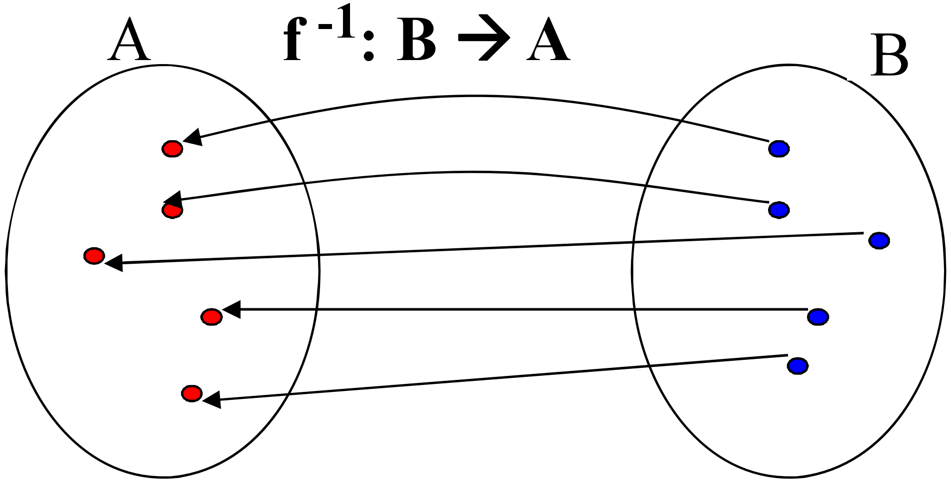


Inverse functions

Definition: Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible.**



f is bijective



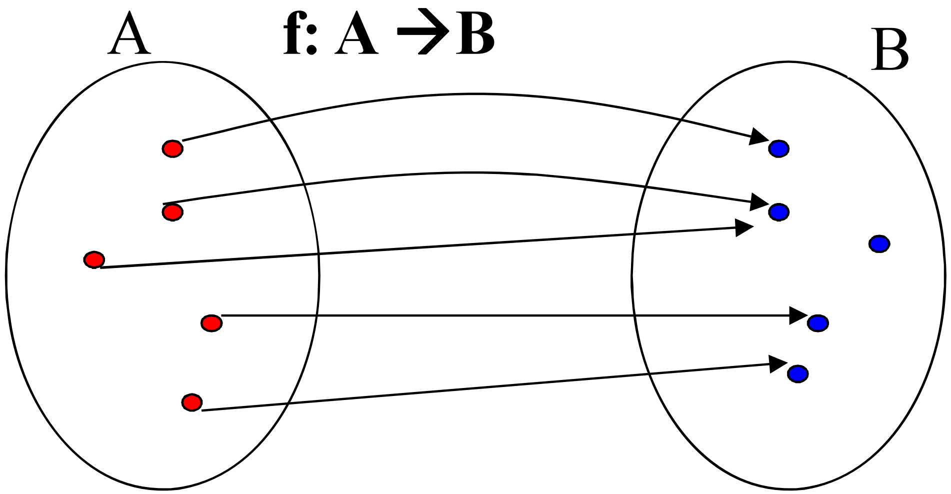
Inverse of f

Inverse functions

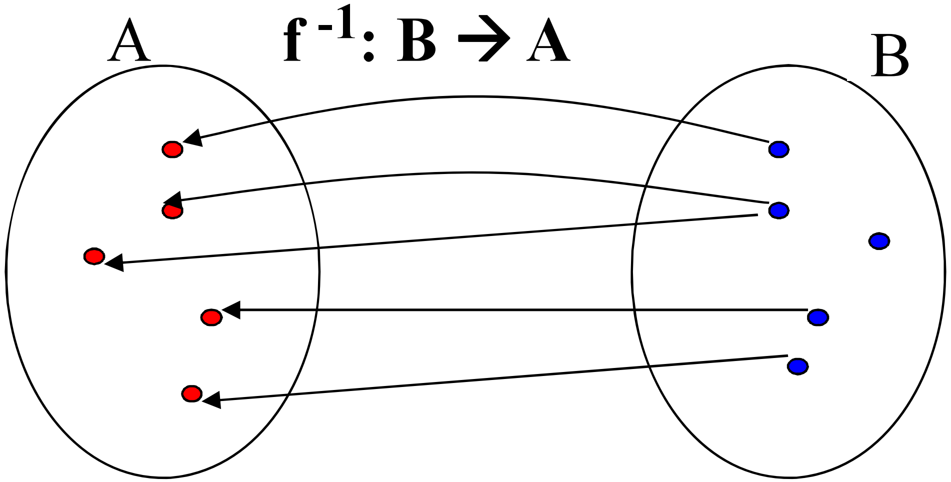
Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

?



f



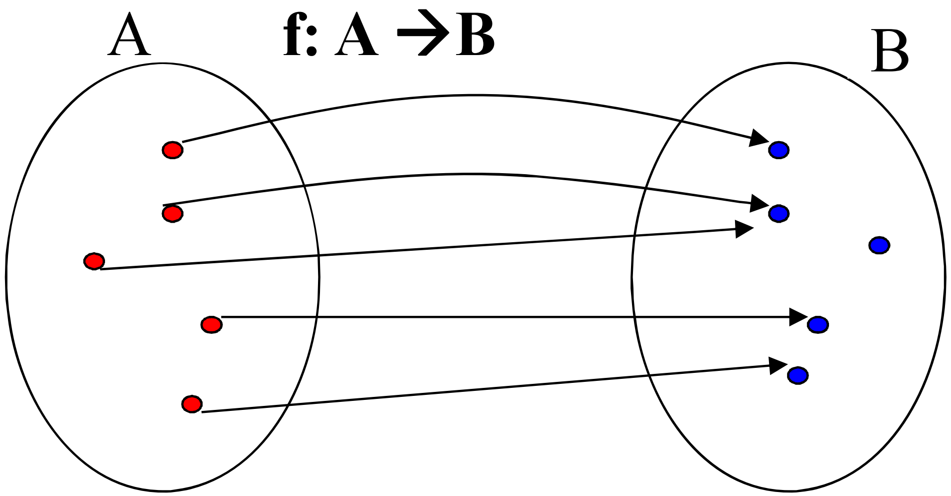
‘Inverse’

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.



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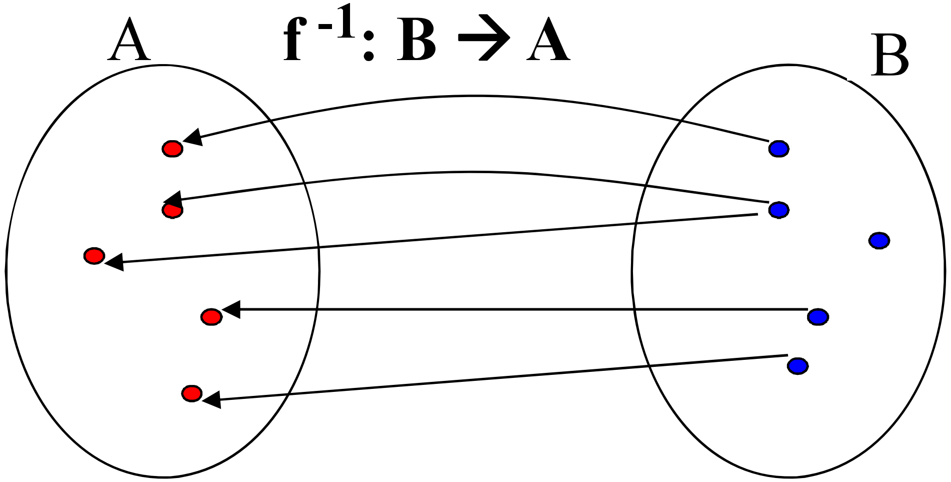


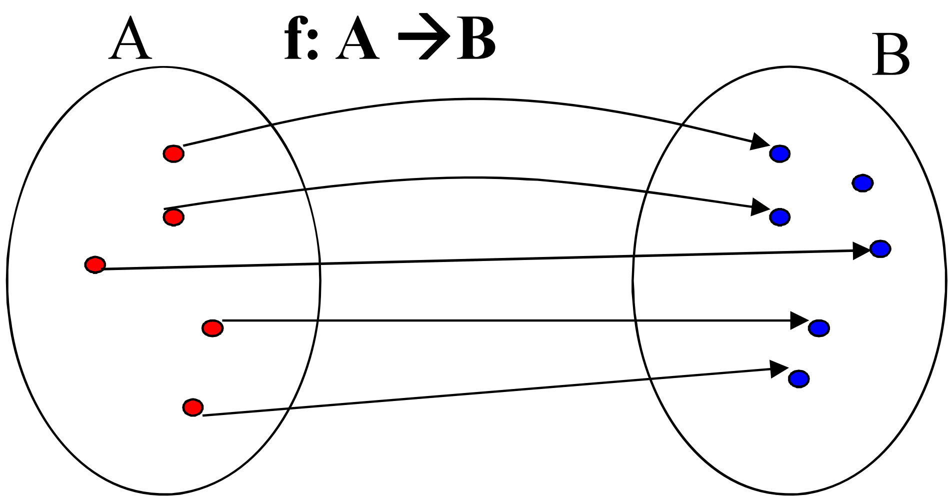
‘Inverse’

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

?



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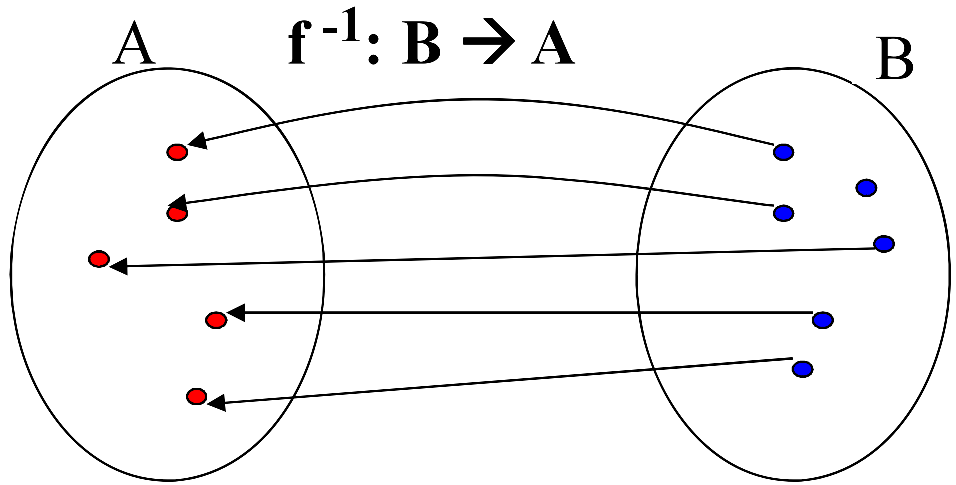


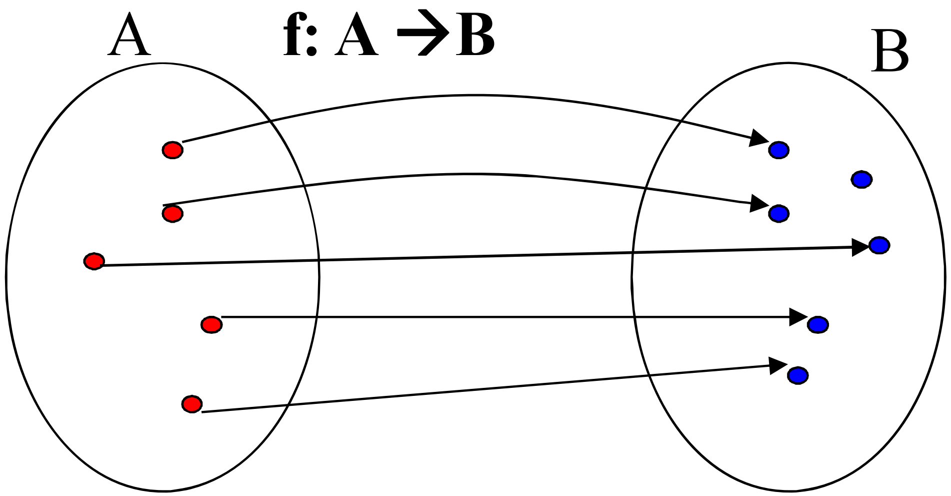
‘Inverse’

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



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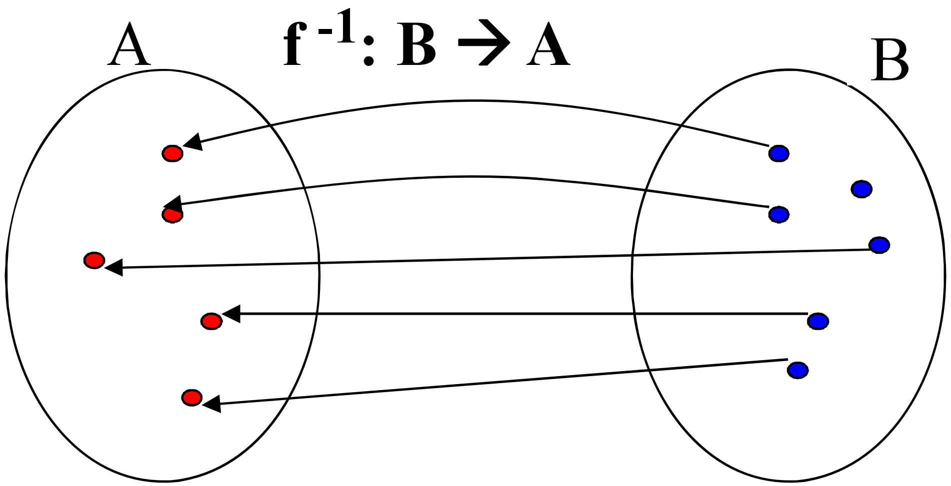


'Inverse'

Inverse functions

Example 1:

- Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\qquad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\qquad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\qquad$ $i_A^{-1}(3) = 3$

- Therefore, the inverse function of i_A is i_A.

Inverse functions

Example 2:

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x$

$\Rightarrow (y+1)/2 = x$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- $\mathbf{g(3)} = ..$

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$$y = 2x - 1 \;\Rightarrow\; y + 1 = 2x$$
$$\Rightarrow (y+1)/2 = x$$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- $\mathbf{g}(3) = 2*3 - 1 = \mathbf{5}$
- $\mathbf{g^{-1}}\ (5) =$

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$$y = 2x - 1 \Rightarrow y + 1 = 2x$$
$$\Rightarrow (y+1)/2 = x$$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- $\mathbf{g}(\mathbf{3}) = 2*3 - 1 = \mathbf{5}$
- $\mathbf{g^{-1}}\ (\mathbf{5}) = (5+1)/2 = \mathbf{3}$
- $g(\mathbf{10}) =$

Inverse functions

Example 2:

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$$y = 2x - 1 \Rightarrow y + 1 = 2x$$

$$\Rightarrow (y+1)/2 = x$$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

Test the correctness of inverse:

- $\mathbf{g(3)} = 2*3 - 1 = \mathbf{5}$
- $\mathbf{g^{-1}\ (5)} = (5+1)/2 = \mathbf{3}$
- $g(\mathbf{10}) = 2*10 - 1 = \mathbf{19}$
- $g^{-1}\ (19) =$

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$$y = 2x - 1 \Rightarrow y + 1 = 2x$$
$$\Rightarrow (y+1)/2 = x$$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

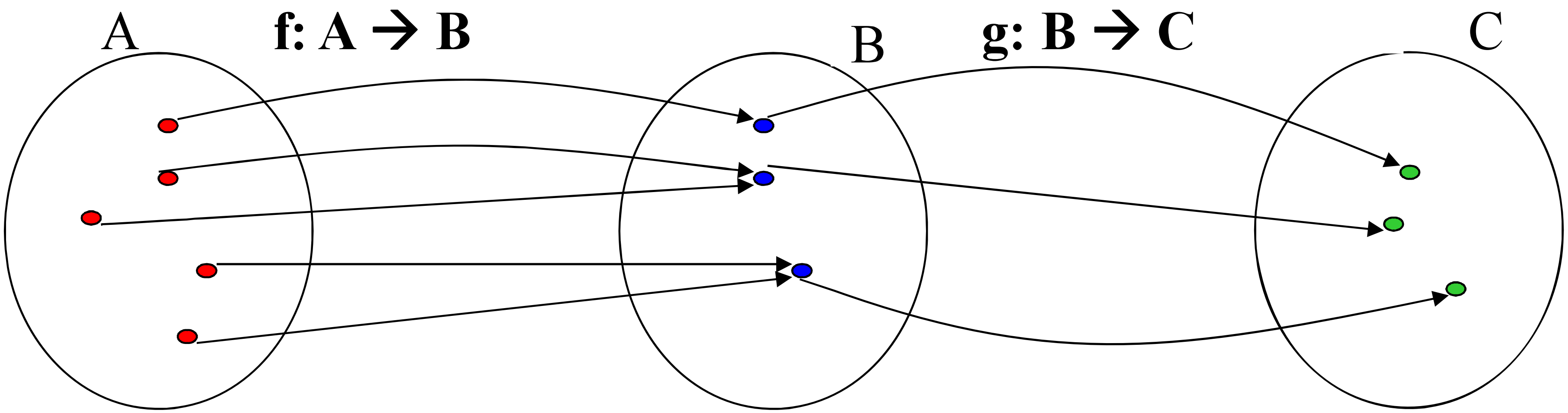
Test the correctness of inverse:

- $\mathbf{g}(\mathbf{3}) = 2*3 - 1 = \mathbf{5}$
- $\mathbf{g^{-1}} (\mathbf{5}) = (5+1)/2 = \mathbf{3}$
- $g(\mathbf{10}) = 2*10 - 1 = \mathbf{19}$
- $g^{-1} (\mathbf{19}) = (19+1)/2 = \mathbf{10}$.

Composition of functions

Definition: Let f be a function from set A to set B and let g be a function from set B to set C. The **composition of the functions g and f**, denoted by g ○ f is defined by

- **(g ○ f)(a) = g(f(a)).**



Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \to A,$ $\quad$ $f: A \to B$

$1 \to 3$ $\quad$ $1 \to b$

$2 \to 1$ $\quad$ $2 \to a$

$3 \to 2$ $\quad$ $3 \to d$

$f \circ g : A \to B:$

- $1 \to$

Composition of functions

Example 1:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

$g : A \to A,$	$f: A \to B$
$1 \to 3$	$1 \to b$
$2 \to 1$	$2 \to a$
$3 \to 2$	$3 \to d$

$f \circ g : A \to B$:

- $1 \to d$
- $2 \to$

Composition of functions

Example 1:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

$g : A \to A$, $\quad$ $f: A \to B$

$1 \to 3$ $\quad$ $1 \to b$

$2 \to 1$ $\quad$ $2 \to a$

$3 \to 2$ $\quad$ $3 \to d$

$f \circ g : A \to B$:

- $1 \to d$
- $2 \to b$
- $3 \to$

Composition of functions

Example 1:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

$g : A \to A$,	$f: A \to B$
$1 \to 3$	$1 \to b$
$2 \to 1$	$2 \to a$
$3 \to 2$	$3 \to d$

$f \circ g : A \to B$:

- $1 \to d$
- $2 \to b$
- $3 \to a$

Composition of functions

Example 2:

- Let f and g be two functions from Z to Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$
- $(f \circ g)(x) \quad = \quad f(g(x))$

  $= \quad f(x^2)$

  $= \quad 2(x^2)$
- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) \quad = \quad ?$

Composition of functions

Example 2:

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- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$
- $(f \circ g)(x) \quad = \quad f(g(x))$

  $= \quad f(x^2)$

  $= \quad 2(x^2)$
- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) \quad = \quad g(f(x))$

  $= \quad g(2x)$

  $= \quad (2x)^2$

  $= \quad 4x^2$

Note that the order of the function composition matters

Composition of functions

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x – 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$$
\begin{aligned}
&= f(\ (x+1)/2\ ) \\
&= 2(\ (x+1)/2\ ) - 1 \\
&= (x+1) - 1 \\
&= x
\end{aligned}
$$

Composition of functions

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \to \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$

  $= f((x+1)/2)$

  $= 2((x+1)/2) - 1$

  $= (x+1) - 1$

  $= x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$

  $= f^{-1}(2x - 1)$

  $= (2x)/2$

  $= x$

Some functions

Definitions:

- The **floor function** assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)$ such that $1! = 1$