East West University, CSE 106

Notes on Discrete Mathematics

Instructor: Dr. Mohammad Salah Uddin

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1 Discrete Mathematics

1.1 Discrete Mathematics

Discrete mathematics is a branch of mathematics that studies mathematical structures that are discrete, meaning made up of distinct, separate values or elements, rather than continuous. It includes topics such as combinatorics, graph theory, number theory, and algorithms.

In contrast to continuous mathematics, which deals with objects that can take on any value in a continuous range, discrete mathematics is concerned with objects that have only a finite or countable number of values. This makes discrete mathematics useful in the study of computer science, as many problems in computer science can be modeled as discrete structures.

Some of the important areas in discrete mathematics are:

- Combinatorics: The study of counting and arrangement of objects, such as permutations and combinations.
- Graph theory: The study of graphs, networks, and their properties, such as connectivity and path length.
- Number theory: The study of the properties of numbers, such as prime numbers and divisibility.
- Algorithms: The study of how to solve problems by step-by-step procedures, such as sorting and searching algorithms.

Discrete mathematics is essential for many areas of computer science, including computer algorithms, computer architecture, and the theory of computation. It is also used in areas such as operations research, coding theory, and cryptography.

2 Propositional Logic

2.1 Propositional logic

Propositional logic, also known as sentential logic or proposition logic, is a branch of mathematical logic that deals with manipulating and analyzing propositions or statements. In propositional logic, propositions are used to express assertions about the world and are either true or false.

The basic building blocks of propositional logic are propositions, connectives, and logical operators. Propositions are statements that can be evaluated as true or false. Connectives are symbols that are used to connect propositions and form more complex statements, such as "and," "or," "not," "implies," and "if and only if." Logical operators are symbols that can be used to manipulate propositions and connectives to form more complex expressions.

The rules and principles of propositional logic can be used to determine the truth or falsity of statements and infer new statements based on given information. This makes propositional logic a fundamental tool in many areas of mathematics, computer science, philosophy, and artificial intelligence.

2.1.1 What is proposition

A proposition is a statement that can be evaluated as true or false. In propositional logic, propositions serve as the basic building blocks for expressing assertions about the world. Propositions are used to make claims about reality and can be used to convey information, express beliefs, or ask questions.

Examples of propositions include "The sky is blue," "All dogs are mammals," "2 + 2 = 4," "John is taller than Jane," and "Today is Monday."

In propositional logic, the focus is not on the actual truth values of propositions but rather on the logical relationships between them. The rules and principles of propositional logic can be used to determine the truth or falsity of statements, as well as to infer new statements based on given information. This makes propositional logic a fundamental tool in many areas of mathematics, computer science, philosophy, and artificial intelligence. Here are some more examples of propositions:

1. The sky is blue.
2. All dogs are mammals.
3. 2 + 2 = 4.

4. John is taller than Jane.
5. Today is Monday.
6. The number of planets in our solar system is 8.
7. Grass is green.
8. Rome is the capital of Italy.
9. The moon is a natural satellite of the Earth.
10. Water freezes at 0 degrees Celsius.
11. The sun rises in the east.
12. All birds can fly.
13. 5 x 5 = 25.
14. Paris is the capital of France.
15. Apples are a type of fruit.
16. The Earth rotates around the sun.
17. Elephants are the largest land animals.
18. The Pythagorean theorem states that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.
19. Cats have sharp claws.
20. The shortest day of the year is December 21st.
21. Mount Everest is the highest mountain in the world.
22. Tigers are carnivores.
23. Humans have ten fingers.
24. Jupiter is the largest planet in our solar system.
25. Fish live in water.
26. London is the capital of England.
27. The speed of light is constant.
28. Dogs have a keen sense of smell.
29. Plants need sunlight to grow.
30. The capital of Australia is Canberra.
31. The Sahara is the largest hot desert in the world.
32. Lions are the king of the jungle.

33. The equator runs through the middle of the Earth.
34. The moon is a celestial body.
35. The Amazon is the largest river in the world.
36. Kangaroos are native to Australia.
37. Snakes are reptiles.
38. The Earth has a magnetic field.
39. The capital of China is Beijing.
40. Vultures are scavengers.
41. The Atlantic Ocean is the second largest ocean in the world.
42. The human brain is the control center of the body.
43. The capital of Germany is Berlin.
44. Sharks are carnivorous fish.
45. The Earth has one natural satellite, the moon.
46. Birds have feathers.
47. The capital of Russia is Moscow.
48. The longest river in the world is the Nile.
49. Ants are social insects.
50. The Earth has four seasons.
51. The capital of India is New Delhi.
52. Horses are domesticated animals.
53. The Earth has a tilted axis.
54. The capital of Canada is Ottawa.
55. The Pacific Ocean is the largest ocean in the world.
56. Butterflies undergo metamorphosis.
57. The capital of Brazil is Brasília.
58. The Great Barrier Reef is the largest coral reef system in the world.
59. Chimpanzees are highly intelligent primates.
60. The Earth has a molten core.
61. The capital of South Africa is Pretoria.

62. Giraffes have long necks.
63. The Earth has a thin atmosphere.
64. The capital of Argentina is Buenos Aires.
65. The Himalayas are the highest mountain range in the world.
66. Squirrels gather food for the winter.
67. The Earth has a strong gravitational pull.
68. The capital of Mexico is Mexico City.
69. Crocodiles are reptiles.
70. The Earth has a unique biosphere.

2.1.2 Why a question is not a proposition?

A proposition is a statement that is either true or false, but not both. It is a declarative sentence that can be either verified or falsified through evidence or reasoning. In other words, a proposition is a type of statement that can be considered as the building block of logical arguments.

On the other hand, a question is not a proposition because it is not a statement that can be verified or falsified. A question is an inquiry that seeks information or clarification, and it typically begins with words such as "what," "when," "where," "who," "why," or "how." A question does not make a definite claim, and it does not have a truth value.

For example, the statement "The sky is blue" is a proposition because it can be verified as true or false through observation. On the other hand, the question "What color is the sky?" is not a proposition because it does not make a definite claim and it seeks information rather than asserting a truth value.

The key difference between a proposition and a question is that a proposition is a statement that can be verified or falsified, while a question is an inquiry that seeks information and does not have a truth value.

2.1.3 Propositional logic in Computer Science

Propositional logic is a fundamental concept in computer science and plays an important role in many areas of the field. Here are some of the reasons for the necessity of propositional logic in computer science:

1. **Theoretical foundations:** Propositional logic provides a mathematical foundation for the study of algorithms, computability, and complexity. It helps to establish a formal framework for reasoning about algorithms and computing systems.

2. **Design of computer hardware:** Propositional logic is used to design digital circuits and computer hardware. Logical operations such as AND, OR, NOT, and XOR can be implemented as gate-level circuits, and the design of these circuits is based on the principles of propositional logic.
3. **Compiler design:** Compilers, which translate high-level programming languages into machine code, use propositional logic to perform various tasks, such as type checking, symbol table management, and code generation.
4. **Artificial intelligence:** Propositional logic is a fundamental concept in artificial intelligence and is used to represent and reason about knowledge in many applications, such as expert systems, natural language processing, and planning systems.
5. **Verification and validation:** Propositional logic is used to formally verify and validate the correctness of computer systems and algorithms. For example, model checking, a technique for automatically checking the correctness of finite-state systems, is based on propositional logic.

In summary, propositional logic is a crucial tool in computer science for reasoning about algorithms, hardware, software, and knowledge representation.

2.2 Logical Connectives

Logical connectives, also known as logical operators, are symbols or words used to connect propositions in a logical expression. They are used to combine simple propositions into more complex statements and to express the relationships between propositions. The most common logical connectives are:

1. **Negation (NOT):** A negation, represented by the symbol "$\neg$" or the word "not," is used to negate or reverse the truth value of a proposition. For example, the statement "It is not the case that John is tall" is true if "John is tall" is false, and vice versa.
2. **Conjunction (AND):** A conjunction, represented by the symbol "$\land$" or the word "and," is used to connect two propositions in such a way that the resulting statement is true only if both of the original propositions are true. For example, the statement "John is tall and Jane is short" is true if and only if both "John is tall" and "Jane is short" are true.
3. **Disjunction (OR):** A disjunction, represented by the symbol "$\lor$" or the word "or," is used to connect two propositions in such a way that the resulting statement is true if either one of the original propositions is true. For example, the statement "John is tall or Jane is short" is true if either "John is tall" or "Jane is short" is true.
4. **Implication (IF-THEN):** An implication, represented by the symbol "$\rightarrow$" or the phrase "if-then," is used to express a relationship between two propositions in which the truth of the first proposition (the antecedent) implies the truth of the second proposition (the consequent). For example, the statement "If it rains, then the streets will be wet" means

Connective	Symbol	Statement
and	$\wedge$	Conjunction
or	$\vee$	Disjunction (inclusive or)
or	$\oplus, \veebar$	Exclusive or
not	$\sim, \neg, \overline{p}$	Negation
if...then	$\rightarrow, \Rightarrow$	Implication (conditional statement)
if and only if	$\leftrightarrow, \Leftrightarrow$	Biconditional

Figure 2.1: Logical Connective symbols(source: google photos).

that if the antecedent "it rains" is true, then the consequent "the streets will be wet" must also be true.

5. **Bi-Implication (IF AND ONLY IF):** A bi-implication, represented by the symbol "↔" or the phrase "if and only if," is used to express a relationship between two propositions in which both propositions must be true or false together. For example, the statement "John is tall if and only if Jane is short" means that "John is tall" and "Jane is short" must have the same truth value.

6. **Exclusive OR (XOR):** logical connective represents a logical operation that outputs true only when one and only one of its inputs is true. It can also be represented by the symbol "⊕".

   Here are some examples of using XOR:

   - In digital circuit design, XOR can be used to compare two binary signals and produce a high output only when the signals are different.
   - In cryptography, XOR can be used as a simple encryption technique by XORing a plaintext message with a secret key to produce the ciphertext. The ciphertext can be decrypted by XORing it with the same key.
   - XOR can be used in error detection and correction. For example, in a binary code, the parity bit can be generated by XORing all the bits in a data word. If the parity bit is incorrect, it indicates that there is an error in the data word.
   - XOR can be used in Boolean logic operations, where it is used to implement logic gates that perform the XOR operation.
   - XOR can be used to toggle a binary value. For example, if we XOR a binary number

with 1, we will get the complement of that number.

These logical connectives can be used to create more complex statements and to reason about the relationships between propositions. Propositional logic provides rules for determining the truth or falsity of statements that use logical connectives, and it can be used to analyze the structure and relationships of statements in natural languages, as well as in formal mathematical and computational systems.

The **contrapositive** of a proposition is a logical equivalent of the original proposition that can be derived by negating both the antecedent and the consequent of an implication and then switching the order of the terms. The contrapositive of a proposition "if p then q" is "if not q then not p".
The contrapositive of a proposition "$p \to q$" is represented as "$\neg q \to \neg p$".

For example, consider the proposition "if it rains, then the roads will be wet". The contrapositive of this proposition is "if the roads are not wet, then it did not rain". This contrapositive is logically equivalent to the original proposition. Both the original proposition and its contrapositive have the same truth value, meaning that if one is true, then so is the other, and if one is false, then so is the other.

The **converse** of a proposition is the reverse of the original proposition. Given a proposition "$p \to q$", the converse of the proposition would be "$q \to p$". In other words, the converse changes the direction of the implication from the original proposition.

For example, the original proposition "If it rains, then the streets will be wet" has the converse "If the streets are wet, then it must have rained". It is important to note that the truth value of the converse of a proposition is not necessarily the same as the truth value of the original proposition.

The **inverse** of a proposition is the negation of the antecedent and consequent of the original proposition. Given a proposition "$p \to q$", the inverse of the proposition would be "$\neg p \to \neg$q". In other words, the inverse of a proposition takes the negation of both the antecedent and consequent of the original proposition.

For example, the original proposition "If it rains, then the streets will be wet" has the inverse "If it does not rain, then the streets will not be wet". It is important to note that the truth value of the inverse of a proposition is not necessarily the same as the truth value of the original proposition.

2.3 Truth Table

A truth table is a tool used in logic and mathematics to determine the truth values of complex propositions based on the truth values of their component propositions. A truth table lists all possible combinations of truth values for the component (elementary) propositions, along with the resulting truth value of the whole proposition.

For example, consider the proposition "P ∧ Q," where "P" and "Q" are two propositions. A truth table for this proposition would have two columns, one for "P" and one for "Q," and four rows corresponding to the four possible combinations of truth values for "P" and "Q":

The truth value of the proposition "P ∧ Q" is given in the last column of the truth table. The truth value is "T" (true) only when both "P" and "Q" are true, and it is "F" (false) in all other cases.

Truth tables can be used to evaluate complex propositions involving multiple logical connectives, such as conjunction, disjunction, negation, implication, and bi-implication. They are also used to test the validity of arguments, to determine the equivalence of propositions, and to simplify complex expressions.

In computer science, truth tables are used in the design and analysis of digital circuits and in the development of algorithms and computer programs. They can also be used to model decision-making processes and to analyze the behavior of systems in artificial intelligence and robotics. Here are the truth tables for the six logical connectives:

Negation (NOT) The truth table for the negation of a proposition "P" is given below:

P	¬ P
T	F
F	F

The truth value of "$\neg P$" is "F" (false) if "P" is true, and "T" (true) if "P" is false. **Conjunction (AND)** The truth table for the conjunction of two propositions "P" and "Q" is given below:

P	Q	P $\land Q$
T	T	T
T	F	F
F	T	F
F	F	F

The truth value of "P ∧ Q" is "T" (true) if and only if both "P" and "Q" are true.

Disjunction (OR) The truth table for the disjunction of two propositions "P" and "Q" is given below:

P	Q	P $\lor Q$
T	T	T
T	F	T
F	T	T
F	F	F

The truth value of "P ∨ Q" is "T" (true) if either "P" or "Q" or both are true.

Implication (IF-THEN) The truth table for the implication of two propositions "P" and "Q" is given below:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth value of "P → Q" is "T" (true) if either "P" is false or "Q" is true, or if both "P" and "Q" are true.

Bi-Implication (IF AND ONLY IF) The truth table for the bi-implication of two propositions "P" and "Q" is given below:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The truth value of "P ↔ Q" is "T" (true) if both "P" and "Q" have the same truth value, and "F" (false) if "P" and "Q" have different truth values.

Exclusive or (XOR) logical connective is a binary operator that evaluates to true if exactly one of its operands is true and false otherwise. The symbol for XOR is typically written as ⊕. The truth table for XOR is as follows:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

In the truth table, the result of P ⊕ Q is true if either P is true and Q is false, or if P is false and Q is true. If both P and Q are either true or false, the result of P ⊕ Q is false.

In some applications, XOR is used as a means of detecting errors in digital data transmission, since a change in the value of just one of the input propositions will result in a change in the truth value of the XOR output. XOR is also used in cryptography and data compression, as well as in the design of digital circuits and computer algorithms.

2.3.1 Truth table construction

To construct a truth table, we first need to identify the propositions involved in the argument. Each proposition is assigned a letter, such as "p" or "q". The truth values of the propositions are then listed in columns, with each row representing a different combination of truth values. The truth values can be represented as either T (true) or F (false).

Here is an example of constructing a truth table for the argument "$p \land q \rightarrow r$". The propositions are "p", "q", and "r", and the logical connectives are "and" (∧) and "implies" (→).

p	q	r	$p \wedge q$	$p \wedge q \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

2.4 Examples of compound proposition

Implication using **"whenever":**

1. "Whenever it rains, the streets get wet." This can be translated into a logical implication as: "If it rains, then the streets get wet."
2. "Whenever John studies, he gets good grades." This can be translated into a logical implication as: "If John studies, then he gets good grades."
3. "Whenever the sun rises, it's a new day." This can be translated into a logical implication as: "If the sun rises, then it's a new day."

Implication using **"sufficient":**

1. "Getting enough sleep is sufficient for feeling rested." This can be translated into a logical implication as: "If you get enough sleep, then you will feel rested."
2. "Having a car is sufficient for being able to drive." This can be translated into a logical implication as: "If you have a car, then you can drive."
3. "Having a degree is sufficient for finding a job." This can be translated into a logical implication as: "If you have a degree, then you can find a job."

Implication using **"unless":**

1. "You will fail the exam unless you study." This can be translated into a logical implication as: "If you don't study, then you will fail the exam."
2. "You will be late unless you leave now." This can be translated into a logical implication as: "If you don't leave now, then you will be late."
3. "The plant will die unless it gets watered." This can be translated into a logical implication as: "If the plant doesn't get watered, then it will die."

Implication using **"provided that":**

1. "I'll help you move provided that you help me paint my room." This can be translated into a logical implication as: "If you help me paint my room, then I'll help you move."

2. "You can borrow my car provided that you fill up the gas tank." This can be translated into a logical implication as: "If you fill up the gas tank, then you can borrow my car."
3. "You can have the cake provided that you share with everyone else." This can be translated into a logical implication as: "If you share with everyone else, then you can have the cake."

Implication using the phrase **"follows from":**

1. "The conclusion follows from the premises." This can be translated into a logical implication as: "If the premises are true, then the conclusion is true."
2. "It follows from the data that the hypothesis is correct." This can be translated into a logical implication as: "If the data is true, then the hypothesis is correct."
3. "The result follows from the equation." This can be translated into a logical implication as: "If the equation is true, then the result is true."

Here are examples of implication using the phrase **"is necessary for"**:

1. "Exercise is necessary for maintaining good health." This can be translated into a logical implication as: "If you want to maintain good health, then you must exercise."
2. "Good sleep is necessary for being productive." This can be translated into a logical implication as: "If you want to be productive, then you must get good sleep."
3. "Water is necessary for plants to grow." This can be translated into a logical implication as: "If you want the plants to grow, then you must provide water."

Here are examples of implication using the phrase **"a sufficient condition for"**:

1. "A degree from a top university is a sufficient condition for getting a high-paying job." This can be translated into a logical implication as: "If you have a degree from a top university, then you are guaranteed a high-paying job."
2. "A good credit score is a sufficient condition for getting a loan." This can be translated into a logical implication as: "If you have a good credit score, then you are guaranteed a loan."
3. "A high IQ is a sufficient condition for being successful." This can be translated into a logical implication as: "If you have a high IQ, then you are guaranteed success."

The phrase **"a sufficient condition for Q is P"** is commonly used in mathematical logic and means that P (a certain condition or proposition) implies that Q (another condition or proposition) is true. In other words, if P is true, then Q must also be true. This relationship can be represented using the logical symbol "→", where $P \rightarrow Q$ means "if P, then Q".

Here are some examples:

1. "A sufficient condition for winning the race is having the fastest time." This means that if a runner has the fastest time, then they must win the race.

2. "A sufficient condition for getting accepted into a college is having a high GPA." This means that if a student has a high GPA, then they will be accepted into the college.
3. "A sufficient condition for a circuit to work is having a complete circuit." This means that if a circuit is complete, then it must work.

Examples of **bi-conditional statements:**

1. If it rains, then the roads will be wet, and if the roads are wet, then it must have rained.
2. You can only enter the club if you are 21 years old or older, and if you are 21 years old or older, then you can enter the club.
3. The game will start at 7 PM, and if the game starts at 7 PM, then it is 7 PM.
4. The lights will turn on if you switch the switch, and if the lights turn on, then you switched the switch.
5. You pass the test if you score 80% or higher, and if you score 80% or higher, then you pass the test.
6. x is even $\Leftrightarrow$ x is divisible by 2
7. a is a multiple of b $\Leftrightarrow$ b is a factor of a
8. n is prime $\Leftrightarrow$ n has exactly two distinct positive divisors
9. x is positive $\Leftrightarrow x > 0$
10. p is a point on line l $\Leftrightarrow$ p lies on l
11. If and only if it rains, the roads will be wet.
12. If and only if you are 21 years old or older, you can enter the club.
13. If and only if the game starts at 7 PM, it is 7 PM.
14. If and only if you switch the switch, the lights will turn on.
15. If and only if you score 80% or higher, you pass the test.

English examples of **Exclusive OR (XOR)** statements:

1. You can either take the train or the bus, but not both.
2. You are either a cat person or a dog person, but not both.
3. The game is either on Sunday or on Monday, but not both.
4. You can either have pizza or sushi for dinner, but not both.
5. You either have a red car or a blue car, but not both.
6. You can only play soccer or basketball, not both.
7. You must choose either apples or oranges, not both.

8. The concert is either on Saturday or on Sunday, but not both.
9. You can either have a sandwich or a salad for lunch, but not both.
10. You can either rent a car or take a taxi, but not both.

2.4.1 Truth value of compound proposition

The truth value of a compound proposition is determined by the truth values of the elementary propositions and the logical connectives used to combine them. For example, consider the following compound proposition:

"It is raining and I have an umbrella"

Let p be the proposition "It is raining" and q be the proposition "I have an umbrella." The compound proposition "p and q" can be represented as $p \land q$. The truth value of $p \land q$ is true if both p and q are true, and false otherwise.

For another example, consider the following compound proposition:

"I will go to the party unless it is raining."

Let p be the proposition "I will go to the party" and q be the proposition "It is raining." The compound proposition "p unless q" can be represented as $p \rightarrow \neg q$. The truth value of $p \rightarrow \neg q$ is true if either p is true and q is false, or p is false. If both p and q are true, the truth value is false.

2.4.2 More Compound Proposition Examples with Truth Values

1. "If it is sunny, then I will go to the beach." - **Truth Value:** True (Sunny → Beach)
2. "I will buy a new phone if and only if it has a good camera." - **Truth Value:** Depends on the camera quality
3. "Either I will study for the exam, or I will fail." - **Truth Value:** True (Studying ∨ Failing)
4. "If it is not raining and I am not tired, I will go for a run." - **Truth Value:** Depends on weather and fatigue/tiredness
5. "I will go to the party, but only if my friends also go." - **Truth Value:** Depends on friends' attendance
6. "I will pass the course only if I score above 70 on the final exam." - **Truth Value:** Depends on exam score
7. "If I wake up early, then I will have time for breakfast." - **Truth Value:** Depends on waking up early
8. "Either I will watch a movie, or I will read a book this evening." - **Truth Value:** Depends on chosen activity

9. "If it is a weekday and I have work, then I will not attend the event." - **Truth Value:** Depends on weekday and work status
10. "I will go shopping if and only if there is a sale." - **Truth Value:** Depends on sale availability
11. "I will not go to the concert unless my favorite band is performing." - **Truth Value:** Depends on favorite band's performance
12. "If the train is delayed, then I will miss my connecting flight." - **Truth Value:** Depends on train delay
13. "I will be happy only if it is a sunny day and I receive good news." - **Truth Value:** Depends on weather and news
14. "Either I will go for a hike, or I will stay at home and relax." - **Truth Value:** Depends on chosen activity
15. "If I finish my work early, then I will have time for a nap." - **Truth Value:** Depends on finishing work early
16. "I will join the gym, but only if they offer a student discount." - **Truth Value:** Depends on student discount
17. "I will not eat dessert unless it is chocolate." - **Truth Value:** Depends on dessert type
18. "If I forget my umbrella, then I will get wet in the rain." - **Truth Value:** Depends on forgetting umbrella
19. "Either I will finish the project by tomorrow, or I will ask for an extension." - **Truth Value:** Depends on project completion
20. "I will attend the conference only if my colleague is also presenting." - **Truth Value:** Depends on colleague's presentation
21. "If the traffic is heavy and I leave late, then I will be late for the meeting." - **Truth Value:** Depends on traffic and departure time
22. "I will take a vacation if and only if my boss approves the time off." - **Truth Value:** Depends on boss approval
23. "Either I will go to the museum, or I will visit the botanical garden." - **Truth Value:** Depends on chosen destination
24. "If I miss the bus, then I will take a taxi to reach the office on time." - **Truth Value:** Depends on missing the bus
25. "I will exercise regularly only if I see positive results." - **Truth Value:** Depends on seeing positive results
26. "If I forget to set an alarm, then I will oversleep." - **Truth Value:** Depends on forgetting to set an alarm

27. "Either I will buy a new car, or I will continue using public transportation." - **Truth Value:** Depends on car purchase decision
28. "I will not order dessert unless the restaurant has my favorite cheesecake." - **Truth Value:** Depends on cheesecake availability
29. "If I win the lottery, then I will travel around the world." - **Truth Value:** Depends on winning the lottery
30. "I will not attend the party unless my best friend is also invited." - **Truth Value:** Depends on best friend's invitation
31. "If it snows heavily, then I will build a snowman." - **Truth Value:** Depends on heavy snowfall
32. "I will buy a house if and only if the interest rates are low." - **Truth Value:** Depends on interest rates
33. "Either I will go for a jog, or I will do yoga at home." - **Truth Value:** Depends on chosen activity
34. "I will not start a new project unless I finish my current tasks." - **Truth Value:** Depends on current task completion
35. "If it is a public holiday, then I will spend the day with family." - **Truth Value:** Depends on public holiday
36. "I will not eat fast food unless it is the only option available." - **Truth Value:** Depends on fast food availability
37. "If the concert tickets are sold out, then I will watch it online." - **Truth Value:** Depends on ticket availability
38. "Either I will cook dinner, or I will order takeout tonight." - **Truth Value:** Depends on chosen option
39. "I will buy a bicycle if and only if I move closer to my workplace." - **Truth Value:** Depends on moving closer
40. "If I miss the bus, then I will wait for the next one." - **Truth Value:** Depends on missing the bus
41. "I will not go to the party unless there is live music." - **Truth Value:** Depends on live music availability
42. "Either I will learn a new language, or I will take a photography course." - **Truth Value:** Depends on chosen learning path
43. "I will attend the workshop only if it covers advanced topics." - **Truth Value:** Depends on workshop content
44. "If it is a hot day, then I will go for a swim." - **Truth Value:** Depends on hot weather

45. "I will not go on a road trip unless I have a reliable car." - **Truth Value:** Depends on car reliability
46. "If I forget my umbrella, then I will get wet in the rain." - **Truth Value:** Depends on forgetting the umbrella
47. "If it is a weekend, then I will go for a bike ride." - **Truth Value:** Depends on whether it's a weekend
48. "I will only eat ice cream if it is a hot summer day." - **Truth Value:** Depends on weather conditions
49. "Either I will finish the novel, or I will start a new one." - **Truth Value:** Depends on reading choices
50. "If I miss the train, then I will take the bus to work." - **Truth Value:** Depends on missing the train
51. "I will attend the meeting only if it is necessary and beneficial." - **Truth Value:** Depends on necessity and benefits
52. "Either I will go to the theater, or I will watch a movie at home." - **Truth Value:** Depends on entertainment choice
53. "If it is a holiday and the weather is good, then I will have a barbecue." - **Truth Value:** Depends on holiday and weather
54. "I will buy a new phone if and only if it has both a good camera and ample storage." - **Truth Value:** Depends on camera quality and storage
55. "Either I will go on a road trip, or I will explore a new city." - **Truth Value:** Depends on travel preference
56. "If I receive a promotion, then I will celebrate with my colleagues." - **Truth Value:** Depends on receiving a promotion
57. "I will exercise regularly only if I can find a workout buddy." - **Truth Value:** Depends on finding a workout buddy
58. "Either I will take a nap, or I will drink a cup of coffee to stay awake." - **Truth Value:** Depends on energy level
59. "If I forget my umbrella, then I will borrow one from a friend." - **Truth Value:** Depends on forgetting the umbrella
60. "I will only go to the amusement park if the roller coaster is operational." - **Truth Value:** Depends on roller coaster status
61. "Either I will learn to play the guitar, or I will take piano lessons." - **Truth Value:** Depends on musical interest

62. "If it is a sale day and I have coupons, then I will go shopping." - **Truth Value:** Depends on sale and coupon availability
63. "I will not stay up late unless there is an important event." - **Truth Value:** Depends on the importance of the event
64. "If I forget to water the plants, then they will wither." - **Truth Value:** Depends on forgetting to water the plants
65. "I will buy a new laptop if and only if it meets both my performance and budget criteria." - **Truth Value:** Depends on laptop specifications and budget
66. "Either I will join a book club, or I will start my own reading group." - **Truth Value:** Depends on social preference

2.5 Contrapositive, Converse, and Inverse Examples

Consider the conditional statement: "If it is raining, then I will stay indoors."

2.5.1 Original Statement:

$$P \to Q$$

Truth Values:

- Scenario 1: It is raining (True), and I stay indoors (True) - Result: True
- Scenario 2: It is not raining (False), and I stay indoors (True) - Result: True
- Scenario 3: It is raining (True), but I do not stay indoors (False) - Result: False
- Scenario 4: It is not raining (False), and I do not stay indoors (False) - Result: True

2.5.2 Contrapositive:

$$\neg Q \to \neg P$$

Truth Values:

- Scenario 1: I don't stay indoors (False), and it is not raining (False) - Result: True
- Scenario 2: I don't stay indoors (False), and it is raining (True) - Result: False
- Scenario 3: I stay indoors (True), and it is not raining (False) - Result: True
- Scenario 4: I stay indoors (True), and it is raining (True) - Result: True

2.5.3 Converse:

$$Q \rightarrow P$$

Truth Values:

- Scenario 1: I stay indoors (True), and it is raining (True) - Result: True
- Scenario 2: I stay indoors (True), and it is not raining (False) - Result: True
- Scenario 3: I don't stay indoors (False), and it is raining (True) - Result: True
- Scenario 4: I don't stay indoors (False), and it is not raining (False) - Result: True

2.5.4 Inverse:

$$\neg P \rightarrow \neg Q$$

Truth Values:

- Scenario 1: It is not raining (False), and I don't stay indoors (False) - Result: True
- Scenario 2: It is raining (True), and I don't stay indoors (False) - Result: False
- Scenario 3: It is not raining (False), and I stay indoors (True) - Result: True
- Scenario 4: It is raining (True), and I stay indoors (True) - Result: True

2.6 Some Mathematical Statements (examples)

2.6.1 Original Statement:

$$P \rightarrow Q$$

"If x is a positive number, then x^2 is positive."

Contrapositive:

$$\neg Q \rightarrow \neg P$$

"If x^2 is not positive, then x is not a positive number."

Converse:

$$Q \rightarrow P$$

"If x^2 is positive, then x is a positive number."

Inverse:

$$\neg P \rightarrow \neg Q$$

"If x is not a positive number, then x^2 is not positive."

2.6.2 Original Statement:

$$R \rightarrow S$$

"If y is an even number, then $y + 2$ is also an even number."

Contrapositive:

$$\neg S \rightarrow \neg R$$

"If $y + 2$ is not an even number, then y is not an even number."

Converse:

$$S \rightarrow R$$

"If $y + 2$ is an even number, then y is also an even number."

Inverse:

$$\neg R \rightarrow \neg S$$

"If y is not an even number, then $y + 2$ is not an even number."

2.6.3 Original Statement:

$$A \rightarrow B$$

"If z is a prime number, then z^2 is not divisible by 4."

Contrapositive:

$$\neg B \rightarrow \neg A$$

"If z^2 is divisible by 4, then z is not a prime number."

Converse:

$$B \rightarrow A$$

"If z^2 is not divisible by 4, then z is a prime number."

Inverse:

$$\neg A \rightarrow \neg B$$

"If z is not a prime number, then z^2 is divisible by 4."

2.7 Bitwise Operation

In computer science, bitwise operations are operations that manipulate individual bits in a binary representation of data. Bitwise operations are used to perform bit-level manipulations of data, such as masking, shifting, and testing individual bits. Here are some common bitwise operations, along with examples:

- **Bitwise AND:** The bitwise AND operation is represented by the symbol "∧". It takes two binary numbers as operands and performs a logical AND operation on each pair of corresponding bits. For example:
  1010
  ∧
  0101
  0000
- **Bitwise OR:** The bitwise OR operation is represented by the symbol "∨". It takes two binary numbers as operands and performs a logical OR operation on each pair of corresponding bits. For example:
  1010 ∨ 0101 = 1111
- **Bitwise XOR:** The bitwise XOR operation is represented by the symbol "⊕". It takes two binary numbers as operands and performs a logical exclusive OR operation on each pair of corresponding bits. For example:
  1010 ⊕ 0101 = 1111
- **Bitwise NOT:** The bitwise NOT operation is represented by the symbol "∼". It takes a single binary number as an operand and performs a logical NOT operation on each bit. For example:
  ∼1010 = 0101
- **Bitwise Left Shift:** The bitwise left shift operation is represented by the symbol "«". It takes a binary number and a shift count as operands and shifts the bits of the binary number to the left by the specified number of positions. For example:
  1010 « 2 = 101000
- **Bitwise Right Shift:** The bitwise right shift operation is represented by the symbol "»". It takes a binary number and a shift count as operands and shifts the bits of the binary number to the right by the specified number of positions. For example:
  1010 » 2 = 0010

These bitwise operations are widely used in computer systems for various purposes, including performing low-level operations on data, implementing efficient algorithms, and solving complex problems in fields such as cryptography and data compression.

2.8 Translating English sentences

In propositional logic, the process of translating English sentences into symbolic representations involves representing the meaning of the sentence as a proposition and using logical connectives, such as "and" (∧), "or" (∨), "not" (¬), "implies" (→), and "if and only if" (↔), to join individual propositions into more complex statements. For example, consider the following English sentence:
"It is raining if and only if the ground is wet."

This can be translated into propositional logic as: $r \leftrightarrow w$
where r represents the proposition "it is raining" and w represents the proposition "the ground is wet." The biconditional symbol ($\leftrightarrow$) states that r and w are logically equivalent, meaning they are both true or both false at the same time. Another example is: "If it is not raining, then the ground is not wet." This can be translated into propositional logic as:
$\neg r \leftrightarrow \neg w$
where $\neg r$ represents the negation of the proposition "it is raining." The implication symbol ($\leftrightarrow$) states that if $\neg r$ is true, then $\neg w$ must also be true.

2.8.1 General rules for translation

1. **Identify the proposition:** Start by identifying the main proposition being made in the sentence, which is usually a statement about the relationship between two or more objects or events.
2. **Assign a propositional variable:** Assign a unique propositional variable to represent each proposition. For example, p, q, r, etc.
3. **Identify logical connectives:** Identify the logical connectives in the sentence, such as "and", "or", "not", "if-then", "if and only if".
4. **Translate logical connectives:** Replace the logical connectives with the appropriate logical symbols. For example, "and" is translated as the symbol $\wedge$, "or" is translated as the symbol $\vee$, "not" is translated as the symbol $\neg$, "if-then" is translated as the symbol $\rightarrow$, and "if and only if" is translated as the symbol $\leftrightarrow$.
5. **Parentheses:** Use parentheses to clarify the order of operations. For example, if there are multiple negations, disjunctions, or conjunctions, use parentheses to indicate the grouping of the sub-formulas.
6. **Simplify:** Finally, simplify the expression by using the rules of propositional logic, such as the associative, commutative, and distributive laws.

Here's an example of translating a natural language sentence into propositional logic with details:
Sentence: "If it is raining, then I will bring an umbrella."

1. Identify the proposition: The main propositions are "I will bring an umbrella." and "It is raining".
2. Assign a propositional variable: Let's assign the proposition "I will bring an umbrella" the propositional variable p, so p := "I will bring an umbrella." and the proposition "It is raining" the propositional variable q, so q := "It is raining."
3. Identify logical connectives: The sentence contains the logical connective "if-then".
4. Translate logical connectives: Replace "if-then" with the symbol $\rightarrow$. So the sentence becomes: "It is raining $\rightarrow$ I will bring an umbrella."

5. Replace propositions with propositional variables: Replace the propositions with the assigned propositional variables. So the sentence becomes: $q \to p$
6. Simplify: No simplification is necessary in this case.

So the final translated propositional logic is: $q \to p$.

2.9 Tautology and contradiction

In propositional logic, a tautology is a formula that is always true regardless of the truth values assigned to its propositions. A tautology can be thought of as a logically valid formula that can never be false.

p	p → p
T	T
F	T

For example, the formula $(p \to p)$ is a tautology because it is true for any truth value assigned to p. If p is true, then $p \to p$ is true because the antecedent (the part before the → symbol) is true and the consequent (the part after the → symbol) is also true. If p is false, then $p \to p$ is also true because the antecedent is false and the consequent is true.

On the other hand, a contradiction is a formula that is always false regardless of the truth values assigned to its propositions. A contradiction can be thought of as a logically invalid formula that can never be true.

p	p ∧ ¬p
T	F
F	F

For example, the formula $(p \land \neg p)$ is a contradiction because it is false for any truth value assigned to p. If p is true, then $p \land \neg p$ is false because the first part of the conjunction (the ∧ symbol) is true and the second part of the conjunction (the ¬ symbol) is false, making the whole formula false. If p is false, then $p \land \neg p$ is also false because the first part of the conjunction is false and the second part of the conjunction is true, making the whole formula false.

2.10 Logical equivalences

Commutativity of ∧:	$p \land q \equiv q \land p$
Commutativity of ∨:	$p \lor q \equiv q \lor p$
Associativity of ∧:	$(p \land q) \land r \equiv p \land (q \land r)$
Associativity of ∨:	$(p \lor q) \lor r \equiv p \lor (q \lor r)$
Distributivity of ∧ over ∨:	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
Distributivity of ∨ over ∧:	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
Idempotence of ∧:	$p \land p \equiv p$
Idempotence of ∨:	$p \lor p \equiv p$
Double negation:	$\neg(\neg p) \equiv p$
De Morgan's laws:	$\neg(p \land q) \equiv \neg p \lor \neg q$
	$\neg(p \lor q) \equiv \neg p \land \neg q$

Logical equivalences, also known as logical identities or tautologies, are statements in propositional logic that are always true, regardless of the truth values of their constituent propositions. In discrete mathematics, understanding these equivalences is essential for simplifying logical expressions, making proofs, and solving problems. Here are some common logical equivalences with explanations:

1. **Double Negation:**

   $$\neg(\neg P) \equiv P$$

   This states that negating a proposition twice is equivalent to the original proposition.

2. **Law of Contrapositive:**

   $$(P \to Q) \equiv (\neg Q \to \neg P)$$

   If the implication "if P, then Q" is true, it is logically equivalent to its contrapositive.

3. **De Morgan's Laws:**

   $$\neg(P \land Q) \equiv (\neg P \lor \neg Q)$$
   $$\neg(P \lor Q) \equiv (\neg P \land \neg Q)$$

   These laws describe how negation distributes over conjunction (AND) and disjunction (OR).

4. **Idempotent Laws:**

   $$(P \land P) \equiv P$$
   $$(P \lor P) \equiv P$$

   These laws state that repeating an operation on a proposition does not change its truth value.

5. **Commutative Laws:**

   $$(P \land Q) \equiv (Q \land P)$$
   $$(P \lor Q) \equiv (Q \lor P)$$

   These laws show that the order of conjunction and disjunction does not affect the result.

6. **Associative Laws:**

   $$(P \land Q) \land R \equiv P \land (Q \land R)$$
   $$(P \lor Q) \lor R \equiv P \lor (Q \lor R)$$

   These laws indicate that grouping propositions with parentheses does not change the outcome.

7. **Distributive Laws:**

$$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$$
$$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$$

   These laws describe how conjunction and disjunction distribute over each other.

8. **Absorption Laws:**

$$P \lor (P \land Q) \equiv P$$
$$P \land (P \lor Q) \equiv P$$

   These laws illustrate how a proposition combined with its own conjunction or disjunction simplifies.

9. **Identity Laws:**

$$P \land \mathbf{T} \equiv P$$
$$P \lor \mathbf{F} \equiv P$$

   These laws show that combining a proposition with a tautology (T) or a contradiction (F) does not change the proposition.

10. **Implication Laws:**

$$P \rightarrow Q \equiv \neg P \lor Q$$
$$P \rightarrow Q \equiv \neg(P \land \neg Q)$$

   These laws provide alternative ways to express implications.

Kindly refer to the book for more details and examples.

Exercises

Exercise 1: Simplify the expression $(P \land \neg Q) \lor (\neg P \land Q)$.

Solution 1: Using the Commutative Law for Disjunction:

$$\begin{aligned}(P \land \neg Q) \lor (\neg P \land Q) &\equiv (\neg Q \land P) \lor (Q \land \neg P) \\ &\equiv (P \land \neg Q) \lor (\neg P \land Q)\end{aligned}$$

So, the expression remains $(P \land \neg Q) \lor (\neg P \land Q)$.

Exercise 2: Simplify the expression $\neg(P \land Q) \land (P \lor \neg Q)$.

Solution 2: Using De Morgan's Laws:

$$\begin{aligned}\neg(P \land Q) \land (P \lor \neg Q) &\equiv (\neg P \lor \neg Q) \land (P \lor \neg Q) \\ &\equiv (\neg P \land P) \lor (\neg P \land \neg Q) \lor (\neg Q \land P) \lor (\neg Q \land \neg Q) \\ &\equiv T \lor (\neg P \land \neg Q) \lor (T) \lor (T) \\ &\equiv T \lor (\neg P \land \neg Q)\end{aligned}$$

So, the simplified expression is $T \lor (\neg P \land \neg Q)$.

Exercise 3: Simplify the expression $\neg(P \lor \neg Q) \lor (P \land \neg Q)$.

Solution 3: Using De Morgan's Laws:

$$\neg(P \lor \neg Q) \lor (P \land \neg Q) \equiv (\neg P \land Q) \lor (P \land \neg Q)$$

So, the simplified expression is $(\neg P \land Q) \lor (P \land \neg Q)$.

Exercise 4: Simplify the expression $\neg(P \land Q) \lor (P \land \neg Q)$.

Solution 4: Using the Distributive Law:

$$\neg(P \land Q) \lor (P \land \neg Q) \equiv (\neg P \lor \neg Q) \lor (P \land \neg Q)$$

So, the simplified expression is $(\neg P \lor \neg Q) \lor (P \land \neg Q)$.

Exercise 5: Simplify the expression $(P \lor Q) \land (\neg P \lor \neg Q)$.

Solution 5: Using the Distributive Law:

$$\begin{aligned}(P \lor Q) \land (\neg P \lor \neg Q) &\equiv (P \land \neg P) \lor (P \land \neg Q) \lor (Q \land \neg P) \lor (Q \land \neg Q) \\ &\equiv T \lor (P \land \neg Q) \lor (Q \land \neg P) \lor T \\ &\equiv (P \land \neg Q) \lor (Q \land \neg P)\end{aligned}$$

So, the simplified expression is $(P \land \neg Q) \lor (Q \land \neg P)$.

Exercise 6: Simplify the expression $\neg(P \lor \neg Q) \lor (P \land \neg Q)$.

Solution 6: Using De Morgan's Laws:

$$\neg(P \lor \neg Q) \lor (P \land \neg Q) \equiv (\neg P \land Q) \lor (P \land \neg Q)$$

So, the simplified expression is $(\neg P \land Q) \lor (P \land \neg Q)$.

Exercise 7: Simplify the expression $(P \to Q) \land (\neg Q \to \neg P)$.

Solution 7: Using the Implication Law:

$$\begin{aligned}(P \to Q) \land (\neg Q \to \neg P) &\equiv (\neg P \lor Q) \land (Q \lor \neg P) \\ &\equiv (\neg P \lor Q) \land (\neg P \lor Q)\end{aligned}$$

So, the simplified expression is $(\neg P \lor Q) \land (\neg P \lor Q)$.

Exercise 8: Simplify the expression $(P \land Q) \lor (\neg P \land Q)$.

Solution 8: Using the Commutative Law for Disjunction:

$$\begin{aligned}(P \land Q) \lor (\neg P \land Q) &\equiv (Q \land P) \lor (Q \land \neg P)\\ &\equiv (P \land Q) \lor (Q \land \neg P)\end{aligned}$$

So, the expression remains $(P \land Q) \lor (Q \land \neg P)$.

Exercise 9: Simplify the expression $\neg(P \land Q) \lor (P \land \neg Q)$.

Solution 9: Using the De Morgan's Laws:

$$\neg(P \land Q) \lor (P \land \neg Q) \equiv (\neg P \lor \neg Q) \lor (P \land \neg Q)$$

So, the simplified expression is $(\neg P \lor \neg Q) \lor (P \land \neg Q)$.

Exercise 10: Simplify the expression $P \land (Q \lor \neg Q)$.

Solution 10: In this expression, $Q \lor \neg Q$ is always true (Tautology), so the entire expression simplifies to just P.

Exercise 11: Simplify the expression $P \land (P \lor Q)$.

Solution 11: This expression represents the Distributive Law, so it remains as $P \land (P \lor Q)$.

Exercise 12: Simplify the expression $\neg(P \lor \neg Q) \lor (P \land \neg Q)$.

Solution 12: Using De Morgan's Laws:

$$\neg(P \lor \neg Q) \lor (P \land \neg Q) \equiv (\neg P \land Q) \lor (P \land \neg Q)$$

So, the simplified expression is $(\neg P \land Q) \lor (P \land \neg Q)$.