Predicate Logic, Discrete Mathematics

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November 11, 2024

Limitations of Propositional Logic

- **Lack of Expressiveness**:
  Propositional logic cannot express statements involving variables or quantifiers. For example, statements like "All humans are mortal" or "Some dogs are friendly" cannot be represented.
- **No Relations Between Objects**:
  It cannot express relationships between multiple entities, such as "John is taller than Sarah."
- **Scalability Issues**:
  Propositional logic needs a separate statement for each fact. When there are many facts to represent, it becomes hard to manage. For example, describing each person in a large group would require many individual statements, making it difficult to work with on a large scale.
- **No Quantifiers**:
  Propositional logic lacks quantifiers like "for all" ($\forall$) and "there exists" ($\exists$). Statements involving generalization or existence

Examples of Limitations in Propositional Logic

Example 1: Representing People in a Group

- To express "Shanto is a student," "Urmi is a student," and "Wasim is a student," propositional logic requires separate statements:

$$\text{Student\_Shanto}, \quad \text{Student\_Urmi}, \quad \text{Student\_Wasim}$$

- With 1000 students, 1000 statements would be needed, making it hard to manage.

Examples of Limitations in Propositional Logic (Contd.)

Example 2: Expressing Universal Truths

- To represent "All dogs are friendly," propositional logic requires a statement for each dog:

$$\text{Friendly\_Dog1}, \quad \text{Friendly\_Dog2}, \quad \text{Friendly\_Dog3}, \ldots$$

- Predicate logic can simplify this as:

$$\forall x\, (\text{Dog}(x) \rightarrow \text{Friendly}(x))$$

Examples of Limitations in Propositional Logic (Contd.)

Example 3: Describing Relationships

- For statements like "John likes Mary" and "Alam likes Bithi," propositional logic needs unique statements for each pair:

$$\text{Likes\_John\_Mary}, \quad \text{Likes\_Alam\_Bithi}$$

- Predicate logic allows us to express "likes" as a relationship:

$$\text{Likes}(x, y)$$

- This approach avoids listing every possible pair individually.

Predicate Logic (First-Order Logic)

Predicate logic overcomes the limitations of propositional logic by introducing:

- **Objects and Predicates**: Statements are constructed using objects (e.g., `John`, `Dog`) and predicates (e.g., `is mortal`, `loves`).
  - Example: `Loves(John, Mary)` means "John loves Mary."
- **Quantifiers**:
  - **Universal Quantifier ($\forall$)**: Used for statements true for all members of a set.

    $$\forall x \, \text{Human}(x) \rightarrow \text{Mortal}(x)$$

    Meaning: "All humans are mortal."
  - **Existential Quantifier ($\exists$)**: Used for statements where at least one member of a set satisfies a condition.

    $$\exists x \, \text{Dog}(x) \wedge \text{Friendly}(x)$$

    Meaning: "There exists a dog that is friendly."

Complex Relationships in Predicate Logic

Predicate logic allows us to express complex relationships between multiple objects, which is not possible in propositional logic. Here are some examples:

- `Taller(John, Mary)` means "John is taller than Mary."
- This relationship involves two objects (John and Mary) and the predicate "Taller."
- `Parent(Alice, Bob)` means "Alice is a parent of Bob."
- `Sibling(Bob, Sarah)` means "Bob and Sarah are siblings."
- Predicate logic allows us to define relationships between family members clearly.
- `Owns(Alam, BookJava)` means "Alam owns java Book."
- `Owns(John, Car)` means "John owns a car."

Relationships in Predicate Logic

- `WorksFor(Emma, CompanyA)` means "Emma works for CompanyA."
- `Manages(Sarah, Emma)` means "Sarah manages Emma."
- `Teaches(ProfSmith, Calculus)` means "Professor Smith teaches Calculus."
- `Studies(StudentX, Calculus)` means "Student X studies Calculus."

Building Blocks of Predicate Logic

Predicate logic, also known as First-Order Logic, consists of several key components that allow it to represent complex statements and relationships. Here are the main building blocks:

- **Objects (Constants)**:
  - Objects represent specific entities or items within a domain.
  - Examples: `John`, `Alice`, `Dog1`, `BookA`
- **Predicates**:
  - Predicates represent properties of objects or relationships between objects.
  - Notation: `Predicate(Object)` or `Predicate(Object1, Object2)`
  - Examples: `Human(John)` means "John is a human"; `Loves(John, Mary)` means "John loves Mary."
- **Variables**:
  - Variables are placeholders that can represent any object in the domain.
  - Notation: Typically denoted by lowercase letters such as `x`, `y`, `z`.
  - Example: In `Loves(x, Mary)`, `x` can represent any person who might love Mary.

Building Blocks of Predicate Logic

- **Quantifiers**:
  - **Universal Quantifier ($\forall$)**: Asserts that a statement is true for all objects in the domain.

    $$\forall x\,(\text{Human}(x) \rightarrow \text{Mortal}(x))$$

    Meaning: "All humans are mortal."
  - **Existential Quantifier ($\exists$)**: Asserts that a statement is true for at least one object in the domain.

    $$\exists x\,(\text{Dog}(x) \wedge \text{Friendly}(x))$$

    Meaning: "There exists a dog that is friendly."
- **Logical Connectives**:
  - Connectives are used to form complex statements.
    - **Conjunction ($\wedge$)**: And
    - **Disjunction ($\vee$)**: Or
    - **Negation ($\neg$)**: Not
    - **Implication ($\rightarrow$)**: If-then
  - Example: `Human(x)` $\wedge$ `Mortal(x)` means "x is both human and mortal."

Domain in Predicate Logic

In predicate logic, the **domain** is the set of all possible objects that variables in predicate statements can represent. The choice of domain affects the truth value of statements, especially when using quantifiers.

- **Definition of Domain**
  - The domain is the collection of objects that variables can refer to in a logical statement.
  - The domain is often specified based on the context, such as "all people," "all students," or "all natural numbers."
- **Example Domains and Their Impact on Truth Values**
  - **Example 1:** Domain = {All people}
    - Predicate: `Loves(x, y)` - "x loves y."
    - Statement: $\forall x \, \exists y \, Loves(x, y)$
    - Meaning: "Every person loves someone."
    - Truth Value: True or false, depending on whether each person in the domain has someone they love.
  - **Example 2:** Domain = {All natural numbers}
    - Predicate: `Even(x)` - "x is an even number."
    - Statement: $\forall x \, (x \geq 0 \rightarrow Even(x) \vee \neg Even(x))$

Identifying Predicates in Predicate Logic

In predicate logic, a **predicate** is an expression that represents a property of objects or a relationship between objects. It takes one or more arguments (objects or variables) and returns true or false.

- **Predicates (Valid)**:
  - `Human(John)` - Represents the property "John is a human."
  - `Loves(Alice, Bob)` - Represents the relationship "Alice loves Bob."
  - `GreaterThan(x, y)` - Represents the relationship "x is greater than y."
  - `Dog(D)` - Represents the property "D is a dog."
- **Not Predicates (Invalid)**:
  - `John` - This is simply a constant representing an object, not a predicate.
  - `Human` - Without any argument, it does not convey a complete meaning (no specific object is described as human).
  - `Loves(Alice)` - Predicates require the correct number of arguments; "Loves" expects two arguments to complete the relationship.
  - `3 + 4` - This is an arithmetic expression; it does not convey a property or relationship that can be true or false.

Understanding Predicates

Predicates represent properties or relationships among objects. A predicate $P(x)$ assigns a truth value (true or false) to each x depending on whether the property holds for x.

- The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

Example:

- Let Student(x) denote a predicate where the universe of discourse is people.
- Student(John) **T** (if John is a student)
- Student(Ann) **T** (if Ann is a student)
- Student(Jane) **F** (if Jane is not a student)

Predicate Logic Example: Prime Numbers

Let $P(x)$ be a predicate representing the statement:

"x is a prime number"

Determine the truth values for each statement below:

- $P(2)$ **T**
- $P(3)$ **T**
- $P(4)$ **F**
- $P(5)$ **T**
- $P(6)$ **F**
- $P(7)$ **T**
- $P(8)$ **F**
- $P(9)$ **F**
- $P(11)$ **T**

Predicate Logic Examples

Let $P(x)$ represent the following statements. Determine the truth values:

1. $P(x)$: "x is an even number"

- $P(1)$ **F**
- $P(2)$ **T**
- $P(3)$ **F**
- $P(4)$ **T**
- $P(5)$ **F**

Predicate Logic Examples

2. $Q(x)$: "x is a multiple of 3"

- $Q(3)$ **T**
- $Q(4)$ **F**
- $Q(6)$ **T**
- $Q(8)$ **F**
- $Q(9)$ **T**

3. $R(x)$: "x is a positive number"

- $R(-2)$ **F**
- $R(0)$ **F**
- $R(1)$ **T**
- $R(5)$ **T**
- $R(-10)$ **F**

Example: Predicate vs. Proposition

Example: Let $Q(x, y)$ denote "$x + 5 > y$"

- Is $Q(x, y)$ a proposition? No! - $Q(x, y)$ depends on the values of x and y and does not have a definite truth value without them. - It is a predicate, not a proposition.
- Is $Q(3, 7)$ a proposition? Yes, it is true.
  - Truth Values:
    - $Q(3, 7)$: **T**
    - $Q(1, 6)$: **F**
    - $Q(2, 2)$: **T**
- Is $Q(3, y)$ a proposition? No! We cannot say if it is true or false without a specific value for y.

Compound Statements and Logical Connectives

Compound statements combine simpler statements using logical connectives.

Examples:

- Student(Lucy) $\wedge$ Student(Jack)
  - Translation: "Both Lucy and Jack are students"
  - Proposition: **Yes**
- Country(Dhaka) $\vee$ River(Dhaka)
  - Translation: "Dhaka is a country or a river"
  - Proposition: **Yes**
- CSE-major(x) $\rightarrow$ Student(x)
  - Translation: "If x is a CSE-major, then x is a student"
  - Proposition: **No** (depends on the value of x)

Compound Statements: More Examples

More examples of compound statements using logical connectives.

- Tall(Alice) $\wedge$ Athlete(Alice)
  - Translation: "Alice is tall and an athlete."
  - Proposition: **Yes**
- Animal(Dog) $\vee$ Plant(Dog)
  - Translation: "A dog is an animal or a plant."
  - Proposition: **Yes**
- Car(x) $\rightarrow$ Vehicle(x)
  - Translation: "If x is a car, then x is a vehicle."
  - Proposition: **No** (depends on the value of x)
- Rainy(yesterday) $\vee$ Sunny(today)
  - Translation: "It was rainy yesterday or sunny today."
  - Proposition: **Yes**

Universal Quantification

The universal quantification of $P(x)$ is the proposition:

"$P(x)$ is true for all values of x in the domain of discourse."

- The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$, meaning "for every x, $P(x)$ is true."

Example:

- Let $P(x)$ denote $x > x - 1$.
- What is the truth value of $\forall x\, P(x)$?
- Assume the universe of discourse of x is all real numbers.
- **Answer:** Since every real number x is greater than $x - 1$, $\forall x\, P(x)$ is true.

Counterexamples

A universal quantifier $\forall x\, P(x)$ claims that $P(x)$ is true for every x in the domain. A single counterexample is enough to disprove it.

Example 1:

- Let $P(x)$ denote "$x^2 \geq x$".
- Assume the domain of x is all real numbers.
- **Counterexample:** If $x = 0.5$, then $0.5^2 = 0.25$, which is not greater than or equal to 0.5.
- Therefore, $\forall x\, P(x)$ is **false**.

Example 2:

- Let $Q(x)$ denote "$x + 1 > x$".
- Assume the domain of x is all real numbers.
- There is no counterexample, since $x + 1$ is always greater than x.
- Therefore, $\forall x\, Q(x)$ is **true**.

Universally Quantified Statements

Example 1: CSE-major(x) $\rightarrow$ Student(x)

- Translation: "If x is a CSE-major, then x is a student."
- Proposition: **No** (depends on the value of x)

Example 2: $\forall x$ (CSE-major(x) $\rightarrow$ Student(x))

- Translation: "For all people, if a person is a CSE-major, then she/he is a student."
- Proposition: **Yes** (the statement holds universally across all values of x)

Existential Quantification

The existential quantification of $P(x)$ is the proposition:

"There exists an element in the domain of discourse such that $P(x)$ is true."

- The notation $\exists x\, P(x)$ denotes the existential quantification of $P(x)$, meaning "there is an x such that $P(x)$ is true."

Example:

- Let $T(x)$ denote $x > 5$, where x is a real number.
- What is the truth value of $\exists x\, T(x)$?
- **Answer:** Since $10 > 5$ is true, there exists an x (e.g., $x = 10$) such that $T(x)$ holds.
- Therefore, $\exists x\, T(x)$ is **true**.

More Examples of Existential Quantification

The existential quantifier $\exists x\, P(x)$ states that there is at least one x in the domain for which $P(x)$ is true.

Example 1:

- Let $P(x)$ denote $x^2 = 4$ with x from real numbers.
- **Truth Value of $\exists x\, P(x)$:** True, since $x = 2$ or $x = -2$ satisfies $x^2 = 4$.

Example 2:

- Let $Q(x)$ denote $x < 0$ where x is from the natural numbers.
- **Truth Value of $\exists x\, Q(x)$:** False, since no natural number is less than 0.

Example 3:

- Let $R(x)$ denote $x + 3 = 7$ with x from the integers.
- **Truth Value of $\exists x\, R(x)$:** True, as $x = 4$ makes $x + 3 = 7$.

Statements About Groups of Objects

Example 1:

- CSE-EWU-graduate(x) $\land$ Honor-student(x)
- Translation: "x is a CSE-EWU graduate and x is an honor student."
- Proposition: **No** (depends on x, cannot be determined for all x)

Example 2:

- $\exists x$ (CSE-EWU-graduate(x) $\land$ Honor-student(x))
- Translation: "There exists a person who is both a CSE-EWU graduate and an honor student."
- Proposition: **Yes** (since we can find such an x)

More Examples: Statements About Groups of Objects

Example 3:

- $\text{Employee}(x) \land \text{Works-at}(x, \text{Company A})$
- Translation: "x is an employee and x works at Company A."
- Proposition: **No** (depends on x, cannot be determined for all x)

Example 4:

- $\exists x\, (\text{Employee}(x) \land \text{Works-at}(x, \text{Company A}))$
- Translation: "There exists a person who is an employee and works at Company A."
- Proposition: **Yes** (since such an x could exist)

Example 5:

- $\forall x\, (\text{Student}(x) \rightarrow \text{Attends}(x, \text{University B}))$
- Translation: "For all x, if x is a student, then x attends University B."
- Proposition: **No** (depends on x, not necessarily true for all x)

Summary of Quantified Statements

When are $\forall x\, P(x)$ and $\exists x\, P(x)$ true or false?

Suppose the universe of discourse consists of $x_1, x_2, \ldots, x_N$. Then:

- $\forall x\, P(x)$ is true if $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_N)$ is true.
- $\exists x\, P(x)$ is true if $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_N)$ is true.

Summary Table:

Statement	When True?
$\forall x\, P(x)$	$P(x)$ true for all x
$\exists x\, P(x)$	There exists some x for which $P(x)$ is true.

Statement	When False?
$\forall x\, P(x)$	There exists an x where $P(x)$ is false.
$\exists x\, P(x)$	$P(x)$ is false for all x.

Translation with Quantifiers

Sentence:

- All EWU students are smart.

Translations:

- **Case 1:** Assume the domain of discourse is EWU students.
  - Translation: $\forall x \, \text{Smart}(x)$
- **Case 2:** Assume the universe of discourse is all students.
  - Translation: $\forall x \, (\text{at}(x, \text{EWU}) \rightarrow \text{Smart}(x))$
- **Case 3:** Assume the universe of discourse is all people.
  - Translation: $\forall x \, (\text{student}(x) \land \text{at}(x, \text{EWU}) \rightarrow \text{Smart}(x))$

Translation with Quantifiers

Sentence:

- Someone at NSU is smart.

Translations:

- **Case 1:** Assume the domain of discourse is all NSU affiliates.
  - Translation: $\exists x \, \text{Smart}(x)$
- **Case 2:** Assume the universe of discourse is all people.
  - Translation: $\exists x \, (\text{at}(x, \text{NSU}) \land \text{Smart}(x))$
- **Case 3:** Assume the universe of discourse is all university students.
  - Translation: $\exists x \, (\text{student}(x) \land \text{at}(x, \text{NSU}) \land \text{Smart}(x))$

Translation with Quantifiers

Sentence:

- There is a vehicle that is electric.

Translations:

- **Case 1:** Assume the domain of discourse is all electric vehicles.
  - Translation: $\exists x \, \text{ElectricVehicle}(x)$
- **Case 2:** Assume the universe of discourse is all vehicles.
  - Translation: $\exists x \, (\text{Vehicle}(x) \wedge \text{Electric}(x))$
- **Case 3:** Assume the universe of discourse is all type of transportation.
  - Translation: $\exists x \, (\text{Transportation}(x) \wedge \text{Vehicle}(x) \wedge \text{Electric}(x))$

Universal and Existential Statements

Universal Statements:

- **Using Implications:**
  - "All $S(x)$ are $P(x)$"
    - Translation: $\forall x(S(x) \rightarrow P(x))$
  - "No $S(x)$ is $P(x)$"
    - Translation: $\forall x(S(x) \rightarrow \neg P(x))$

Existential Statements:

- **Using Conjunctions:**
  - "Some $S(x)$ are $P(x)$"
    - Translation: $\exists x(S(x) \wedge P(x))$
  - "Some $S(x)$ are not $P(x)$"
    - Translation: $\exists x(S(x) \wedge \neg P(x))$

Nested Quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in predicate logic.

Example:

- Sentence: *"Every real number has its corresponding negative."*
- **Translation:**
  - Assume:
    - A real number is denoted as x and its negative as y.
    - A predicate $P(x, y)$ denotes: $x + y = 0$.
  - Formal Expression: $\forall x \exists y \, P(x, y)$

Nested Quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in predicate logic.

Example:

- Sentence: *"There is a person who loves everybody."*
- **Translation:**
  - Assume:
    - Variables x and y denote people.
    - A predicate $L(x, y)$ denotes: "x loves y".
  - Formal Expression: $\exists x \forall y\, L(x, y)$

Order of Quantifiers

Order of Nested Quantifiers:

- The order of nested quantifiers matters when the quantifiers are of different types.
- $\forall x \exists y\, L(x, y)$ is not the same as $\exists y \forall x\, L(x, y)$.

Example:

- Assume $L(x, y)$ denotes: *"x loves y"*
- $\forall x \exists y\, L(x, y)$:
  - Translation: *"Everybody loves somebody."*
- $\exists y \forall x\, L(x, y)$:
  - Translation: *"There is someone who is loved by everyone."*

The meanings of the two expressions are different due to the order of quantifiers.

Order of Quantifiers (Same Type)

- The order of nested quantifiers does not matter if the quantifiers are of the same type (both universal or both existential).

Example:

- Statement: *"For all x and y, if x is a parent of y, then y is a child of x."*
- Assume:
  - $Parent(x, y)$ denotes: *"x is a parent of y"*
  - $Child(x, y)$ denotes: *"x is a child of y"*
- Two equivalent translations:
  - $\forall x \forall y \, (Parent(x, y) \rightarrow Child(y, x))$
  - $\forall y \forall x \, (Parent(x, y) \rightarrow Child(y, x))$

The order of universal quantifiers ($\forall$) does not affect the meaning in this context.

Translation with Quantifiers

Supose:

- Variables x, y denote people
- $L(x, y)$ denotes "x loves y"

Translations:

- **Everybody loves Raymond:**
  $\forall x \, L(x, \text{Raymond})$
- **Everybody loves somebody:**
  $\forall x \, \exists y \, L(x, y)$
- **There is somebody whom everybody loves:**
  $\exists y \, \forall x \, L(x, y)$
- **There is somebody who Raymond doesn't love:**
  $\exists y \, \neg L(\text{Raymond}, y)$
- **There is somebody whom no one loves:**
  $\exists y \, \forall x \, \neg L(x, y)$

Examples of Translation with Quantifiers

- **Every student likes some teacher.**
  Translation: $\forall x\,(\text{Student}(x) \rightarrow \exists y\,\text{Teacher}(y) \wedge \text{Likes}(x, y))$
  Explanation: For each student, there exists a teacher that the student likes.
- **There is a person who likes all students.**
  Translation: $\exists x\,\forall y\,(\text{Student}(y) \rightarrow \text{Likes}(x, y))$
  Explanation: There exists someone who likes every student.
- **Some students don't like any teacher.**
  Translation: $\exists x\,(\text{Student}(x) \wedge \forall y\,(\text{Teacher}(y) \rightarrow \neg\text{Likes}(x, y)))$
  Explanation: There is a student who dislikes all teachers.

Negation of Quantifiers

English Statement:
Nothing is perfect.
Translation:
$\neg \exists x \, \text{Perfect}(x)$**Alternative Expression:**
Everything is imperfect.
$\forall x \neg \text{Perfect}(x)$

Negation of Quantifiers Example

English Statement:
"There is no student who failed the exam."
Translation:
$\neg \exists x \, \text{Failed}(x)$
Alternatively:
$\forall x \neg \, \text{Failed}(x)$

- **English Statement:** "Everyone passed the exam."
- **Translation:** $\forall x \, \text{Passed}(x)$
- **English Statement:** "There is a student who passed the exam."
- **Translation:** $\exists x \, \text{Passed}(x)$

Negation of Quantifiers

English Statement:
"Nothing is perfect."
Translation:
$\neg \exists x \, \text{Perfect}(x)$

Another way to express the same meaning:
"Everything is imperfect."
Translation:
$\forall x \, \neg \, \text{Perfect}(x)$

$\neg \exists x \, P(x)$ is equivalent to $\forall x \, \neg \, P(x)$

Negation of Quantified Statements (DeMorgan's Laws)

Negation	**Equivalent**
$\neg \exists x \, P(x)$	$\forall x \, \neg P(x)$
$\neg \forall x \, P(x)$	$\exists x \, \neg P(x)$

Negation of Quantified Statements (DeMorgan's Laws)

Negation	**Equivalent**
$\neg\exists x\,(x^2 = 4)$	$\forall x\,(x^2 \neq 4)$
$\neg\forall x\,(x + 2 > 5)$	$\exists x\,(x + 2 \leq 5)$

- $\neg\exists x\,(x^2 = 4)$ means "There does not exist an x such that $x^2 = 4$."
- The negation would be: "For all x, $x^2 \neq 4$."
- $\neg\forall x\,(x + 2 > 5)$ means "It is not true that for all x, $x + 2 > 5$."
- The negation would be: "There exists an x such that $x + 2 \leq 5$."