Relations

Discrete Mathematics

Dr. Mohammad Salah Uddin
Associate Professor
Computer Science & Engineering Department
East West University

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Cartesian Product (Review)

Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_m\}$.
The Cartesian product $A \times B$ is defined by a set of pairs:

$$\{(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_m), \ldots, (a_k, b_m)\}.$$

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary Relations

Definition: Let A and B be two sets. A binary relation from A to B is a subset of the Cartesian product $A \times B$.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.
- We use the notation aRb to denote $(a, b) \in R$ and $a \not R b$ to denote $(a, b) \notin R$. If aRb, we say a is related to b by R.

Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

- Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B? **Yes.**
- Is $Q = \{(1, a), (2, b)\}$ a relation from A to B? **No.**
- Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A? **Yes.**

Representing Relations

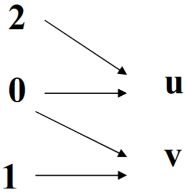
We can graphically represent a binary relation R as follows:

- If aRb, then draw an arrow from a to b: $a \to b$.

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$.
- Note: $R \subseteq A \times B$.

Graph:



Representing Relations

We can represent a binary relation R by a table showing (marking) the ordered pairs of R.

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$.

Table:

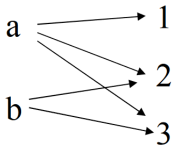
R	u	v
0	$\times$	$\times$
1		$\times$
2	$\times$	

R	u	v
0	1	1
1	0	1
2	1	0

Relations and Functions

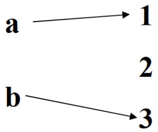
Relations represent one-to-many relationships between elements in A and B.

Example:



Example:

- What is the difference between a relation and a function from A to B? A function defined on sets A, B $A \to B$ assigns to each element in the domain set A exactly one element from B. So it is a special relation.



Relation on the Set

Definition: A relation on the set A is a relation from A to itself.

Example 1:

- Let $A = \{1, 2, 3, 4\}$ and $R_{\text{div}} = \{(a, b) \mid a \text{ divides } b\}$.
- What does R_{div} consist of?
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Table:

R_{div}	1	2	3	4
1	$\times$	$\times$	$\times$	$\times$
2		$\times$		$\times$
3			$\times$	
4				$\times$

Relation on the Set

Example:

- Let $A = \{1, 2, 3, 4\}$.
- Define a relation $aR_{\neq}b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1),$ (3,2),(3,4),(4,1),(4,2),(4,3)}

Table:

$R_{\neq}$	1	2	3	4
1		$\times$	$\times$	$\times$
2	$\times$		$\times$	$\times$
3	$\times$	$\times$		$\times$
4	$\times$	$\times$	$\times$	

Binary Relations

Theorem: The number of binary relations on a set A, where $|A| = n$, is 2^{n^2}.

Proof:

- If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.
- R is a binary relation on A if $R \subseteq A \times A$ (i.e., R is a subset of $A \times A$).
- The number of subsets of a set with k elements is 2^k.
- Therefore, the number of subsets of $A \times A$ is 2^{n^2}.

Properties of Relations

Definition (Reflexive Relation): A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Example 1:

- Assume relation $R_{\text{div}} = \{(a, b) \text{ if } a|b\}$ on $A = \{1, 2, 3, 4\}$.
- Is R_{div} reflexive?
- $R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- Answer: Yes. $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4) \in A$.

Reflexive Relation

Reflexive Relation:

- $R_{\text{div}} = \{(a, b) \text{ if } a|b\}$ on $A = \{1, 2, 3, 4\}$.
- $R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

$$M_{R_{\text{div}}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A relation R is reflexive if and only if M_R has 1 in every position on its main diagonal.

Reflexive Relation

Definition (Reflexive Relation): A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
Example 2:

- Relation R_{fun} on $A = \{1, 2, 3, 4\}$ defined as:
- $R_{\text{fun}} = \{(1, 2), (2, 2), (3, 3)\}$.
- Is R_{fun} reflexive?
- No. It is not reflexive since $(1, 1) \notin R_{\text{fun}}$.

Irreflexive Relation

Definition (Irreflexive Relation): A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every $a \in A$.

Example 1:

- Assume relation $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$.
- Is $R_{\neq}$ irreflexive?
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),$ (3,4),(4,1),(4,2),(4,3)}

- Answer: Yes. Because $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4) \notin R_{\neq}$.

Properties of Relations

Irreflexive Relation

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3),$
(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

$$M_R = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

A relation R is irreflexive if and only if M_R has 0 in every position on its main diagonal.

Properties of Relations

Definition (Irreflexive Relation): A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every $a \in A$.

Example 2:

- R_{fun} on $A = \{1, 2, 3, 4\}$ defined as:
- $R_{\text{fun}} = \{(1, 2), (2, 2), (3, 3)\}$.
- Is R_{fun} irreflexive?
- Answer: No. Because $(2, 2)$ and $(3, 3) \in R_{\text{fun}}$.

Properties of Relations

Definition (Symmetric Relation): A relation R on a set A is called symmetric if for all $a, b \in A$, $(a, b) \in R \rightarrow (b, a) \in R$.
Example 1:

- $R_{\text{div}} = \{(a, b) \text{ if } a|b\}$ on $A = \{1, 2, 3, 4\}$.
- Is R_{div} symmetric?
- $R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- Answer: No. It is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$.

Properties of Relations

Definition (Symmetric Relation): A relation R on a set A is called symmetric if for all $a, b \in A$, $(a, b) \in R \rightarrow (b, a) \in R$.
Example 2:

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$.
- Is $R_{\neq}$ symmetric?
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),$ (3,4),(4,1),(4,2),(4,3)$\}$

- Answer: Yes. If $(a, b) \in R_{\neq} \rightarrow (b, a) \in R_{\neq}$.

Properties of Relations

Symmetric Relation:

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),$ (3,4),(4,1),(4,2),(4,3)}

$$M_R = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

A relation R is symmetric if and only if $m_{ij} = m_{ji}$ for all i, j.

Properties of Relations

Definition (Symmetric Relation): A relation R on a set A is called symmetric if for all $a, b \in A$, $(a, b) \in R \rightarrow (b, a) \in R$.
Example 3:

- Relation R_{fun} on $A = \{1, 2, 3, 4\}$ defined as:
- $R_{\text{fun}} = \{(1, 2), (2, 2), (3, 3)\}$.
- Is R_{fun} symmetric?
- Answer: No. For $(1, 2) \in R_{\text{fun}}$ there is no $(2, 1) \in R_{\text{fun}}$.

Properties of Relations

Definition (Anti-symmetric Relation): A relation on a set A is called anti-symmetric if $[(a, b) \in R$ and $(b, a) \in R] \rightarrow a = b$ where $a, b \in A$.

Example 3:

- Relation R_{fun} on $A = \{1, 2, 3, 4\}$ defined as:
- $R_{\text{fun}} = \{(1, 2), (2, 2), (3, 3)\}$.
- Is R_{fun} anti-symmetric?
- Answer: Yes. It is anti-symmetric.

Properties of Relations

Anti-symmetric Relation

- Relation $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}$

$$M_{R_{\text{fun}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A relation is anti-symmetric if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Properties of Relations

Definition (Transitive Relation): A relation R on a set A is called transitive if $[(a, b) \in R$ and $(b, c) \in R] \rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Example 1:

- $R_{\text{div}} = \{(a, b)$ if $a|b\}$ on $A = \{1, 2, 3, 4\}$.
- $R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.
- Is R_{div} transitive?
- Answer: Yes.

Properties of Relations

Definition (Transitive Relation): A relation R on a set A is called transitive if $[(a, b) \in R$ and $(b, c) \in R] \rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Example 2:

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),$ (3,4),(4,1),(4,2),(4,3)$\}$.

- Is $R_{\neq}$ transitive?
- Answer: No. It is not transitive since $(1, 2) \in R_{\neq}$ and $(2, 1) \in R_{\neq}$, but $(1, 1)$ is not an element of $R_{\neq}$.

Properties of Relations

Definition (Transitive Relation): A relation R on a set A is called transitive if $[(a, b) \in R$ and $(b, c) \in R] \rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Example 3:

- Relation R_{fun} on $A = \{1, 2, 3, 4\}$ defined as:
- $R_{\text{fun}} = \{(1, 2), (2, 2), (3, 3)\}$.
- Is R_{fun} transitive?
- Answer: Yes. It is transitive.

Transitive Relation

Let $A = \{1, 2, 3\}$ and R be the relation on set A defined as $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 3)\}$. The matrix representation of R is:

$$M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that R is transitive.

Transitive Relation (Solution)

We can compute ${M_R}^2 \odot$ as $M_R \cdot M_R$:

$$ {M_R}^2 \odot = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} $$

Since, ${M_R}^2 \odot = M_R$, the relation R is transitive.
Formula: if ${M_R}^2 \odot = M_R$ then the relation is transitive

Equivalence Relations

Definition: An equivalence relation R on a set A is a relation that is reflexive, symmetric, and transitive.

Properties:

- **Reflexive:** $(a, a) \in R$ for all $a \in A$.
- **Symmetric:** $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
- **Transitive:** $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Example: The relation of "equality" is an equivalence relation. For any set A, the relation $R = \{(a, a) \mid a \in A\}$ is an equivalence relation on A.