

Measuring Errors

Why measure errors?

- 1) To determine the accuracy of numerical results.
- 2) To develop stopping criteria for iterative algorithms.

True Error

- Defined as the difference between the true value in a calculation and the approximate value found using a numerical method etc.

True Error = True Value – Approximate Value

Example—True Error

The derivative, $f'(x)$ of a function $f(x)$ can be approximated by the equation,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

If $f(x) = 7e^{0.5x}$ and $h = 0.3$

- a) Find the approximate value of $f'(2)$
- b) True value of $f'(2)$
- c) True error for part (a)

Example (cont.)

Solution:

a) For $x = 2$ and $h = 0.3$

$$\begin{aligned} f'(2) &\approx \frac{f(2+0.3) - f(2)}{0.3} \\ &= \frac{f(2.3) - f(2)}{0.3} \\ &= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3} \\ &= \frac{22.107 - 19.028}{0.3} = 10.263 \end{aligned}$$

Example (cont.)

Solution:

b) The exact value of $f'(2)$ can be found by using our knowledge of differential calculus.

$$\begin{aligned}f(x) &= 7e^{0.5x} \\f'(x) &= 7 \times 0.5 \times e^{0.5x} \\&= 3.5e^{0.5x}\end{aligned}$$

So the true value of $f'(2)$ is

$$\begin{aligned}f'(2) &= 3.5e^{0.5(2)} \\&= 9.5140\end{aligned}$$

True error is calculated as

$$\begin{aligned}E_t &= \text{True Value} - \text{Approximate Value} \\&= 9.5140 - 10.263 = -0.722\end{aligned}$$

Relative True Error

- Defined as the ratio between the true error, and the true value.

$$\text{Relative True Error (} \epsilon_t \text{)} = \frac{\text{True Error}}{\text{True Value}}$$

Example—Relative True Error

Following from the previous example for true error, find the relative true error for $f(x) = 7e^{0.5x}$ at $f'(2)$ with $h = 0.3$

From the previous example,

$$E_t = -0.722$$

Relative True Error is defined as

$$\begin{aligned}\epsilon_t &= \frac{\text{True Error}}{\text{True Value}} \\ &= \frac{-0.722}{9.5140} = -0.075888\end{aligned}$$

as a percentage,

$$\epsilon_t = -0.075888 \times 100\% = -7.5888\%$$

Approximate Error

- What can be done if true values are not known or are very difficult to obtain?
- Approximate error is defined as the difference between the present approximation and the previous approximation.

Approximate Error (E_a) = Present Approximation – Previous Approximation

Example—Approximate Error

For $f(x) = 7e^{0.5x}$ at $x = 2$ find the following,

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) approximate error for the value of $f'(2)$ for part b)

Solution:

a) For $x = 2$ and $h = 0.3$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

Example (cont.)

Solution: (cont.)

$$\begin{aligned} &= \frac{f(2.3) - f(2)}{0.3} \\ &= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3} \\ &= \frac{22.107 - 19.028}{0.3} = 10.263 \end{aligned}$$

b) For $x = 2$ and $h = 0.15$

$$\begin{aligned} f'(2) &\approx \frac{f(2 + 0.15) - f(2)}{0.15} \\ &= \frac{f(2.15) - f(2)}{0.15} \end{aligned}$$

Example (cont.)

Solution: (cont.)

$$\begin{aligned} &= \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15} \\ &= \frac{20.50 - 19.028}{0.15} = 9.8800 \end{aligned}$$

c) So the approximate error, E_a is

$$\begin{aligned} E_a &= \text{Present Approximation} - \text{Previous Approximation} \\ &= 9.8800 - 10.263 \\ &= -0.38300 \end{aligned}$$

Relative Approximate Error

- Defined as the ratio between the approximate error and the present approximation.

$$\text{Relative Approximate Error } (\epsilon_a) = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

Example—Relative Approximate Error

For $f(x) = 7e^{0.5x}$ at $x = 2$, find the relative approximate error using values from $h = 0.3$ and $h = 0.15$

Solution:

From Example 3, the approximate value of $f'(2) = 10.263$ using $h = 0.3$ and $f'(2) = 9.8800$ using $h = 0.15$

$$\begin{aligned} E_a &= \text{Present Approximation} - \text{Previous Approximation} \\ &= 9.8800 - 10.263 \\ &= -0.38300 \end{aligned}$$

Example (cont.)

Solution: (cont.)

$$\begin{aligned}\epsilon_a &= \frac{\text{Approximate Error}}{\text{Present Approximation}} \\ &= \frac{-0.38300}{9.8800} = -0.038765\end{aligned}$$

as a percentage,

$$\epsilon_a = -0.038765 \times 100\% = -3.8765\%$$

Absolute relative approximate errors may also need to be calculated,

$$|\epsilon_a| = |-0.038765| = 0.038765 \text{ or } 3.8765\%$$

How is Absolute Relative Error used as a stopping criterion?

If $|\epsilon_a| \leq \epsilon_s$ where ϵ_s is a pre-specified tolerance, then no further iterations are necessary and the process is stopped.

If at least m significant digits are required to be correct in the final answer, then

$$|\epsilon_a| \leq 0.5 \times 10^{2-m} \%$$

Table of Values

For $f(x) = 7e^{0.5x}$ at $x = 2$ with varying step size, h

h	$f'(2)$	$ \epsilon_a $	m
0.3	10.263	N/A	0
0.15	9.8800	3.877%	1
0.10	9.7558	1.273%	1
0.01	9.5378	2.285%	1
0.001	9.5164	0.2249%	2

THE END