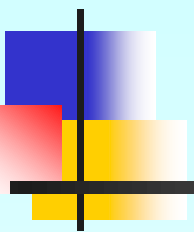
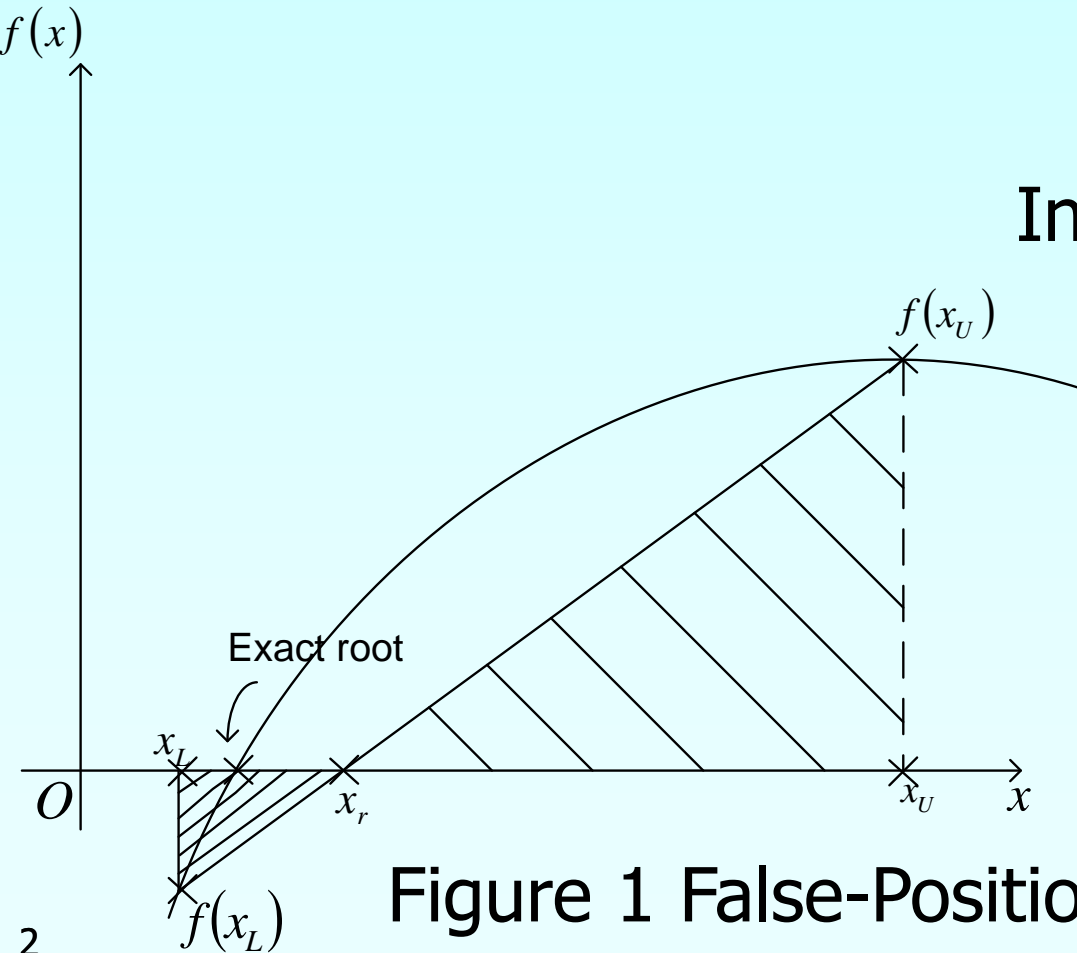


Lecture # 2



False-Position Method of Solving a Nonlinear Equation

Introduction



$$f(x) = 0 \quad (1)$$

In the Bisection method

$$f(x_L) * f(x_U) < 0 \quad (2)$$

$$x_r = \frac{x_L + x_U}{2} \quad (3)$$

Figure 1 False-Position Method



False-Position Method

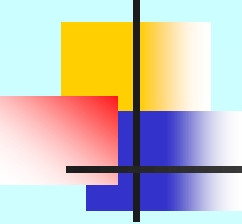
Based on two similar triangles, shown in Figure 1, one gets:

$$\frac{f(x_L)}{x_r - x_L} = \frac{f(x_U)}{x_r - x_U} \quad (4)$$

The signs for both sides of Eq. (4) is consistent, since:

$$f(x_L) < 0; x_r - x_L > 0$$

$$f(x_U) > 0; x_r - x_U < 0$$



From Eq. (4), one obtains

$$\begin{aligned}(x_r - x_L)f(x_U) &= (x_r - x_U)f(x_L) \\ x_U f(x_L) - x_L f(x_U) &= x_r \{f(x_L) - f(x_U)\}\end{aligned}$$

The above equation can be solved to obtain the next predicted root x_r , as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \quad (5)$$



The above equation,

$$x_r = x_U - \frac{f(x_U)\{x_L - x_U\}}{f(x_L) - f(x_U)} \quad (6)$$

or

$$x_r = x_L - \frac{f(x_L)}{\left\{ \frac{f(x_U) - f(x_L)}{x_U - x_L} \right\}} \quad (7)$$



Step-By-Step False-Position Algorithms

1. Choose x_L and x_U as two guesses for the root such that

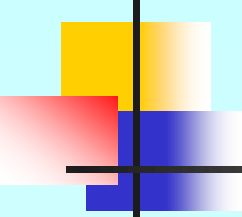
$$f(x_L)f(x_U) < 0$$

2. Estimate the root, $x_m = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$

3. Now check the following

(a) If $f(x_L)f(x_m) < 0$, then the root lies between x_L and x_m ; then $x_L = x_L$ and $x_U = x_m$

(b) If $f(x_L)f(x_m) > 0$, then the root lies between x_m and x_U ; then $x_L = x_m$ and $x_U = x_U$



(c) If $f(x_L)f(x_m) = 0$, then the root is x_m .
Stop the algorithm if this is true.

4. Find the new estimate of the root

$$x_m = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$



where

x_m^{new} = estimated root from present iteration

x_m^{old} = estimated root from previous iteration

5. say $\epsilon_s = 10^{-3} = 0.001$. If $|\epsilon_a| > \epsilon_s$, then go to step 3, else stop the algorithm.

Notes: The False-Position and Bisection algorithms are quite similar. The only difference is the formula used to calculate the new estimate of the root x_m , shown in steps #2 and 4!

Example 1

The floating ball has a specific gravity of 0.6 and has a radius of 5.5cm.

You are asked to find the depth to which the ball is submerged when floating in water.

The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the false-position method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the converged iteration.



Solution

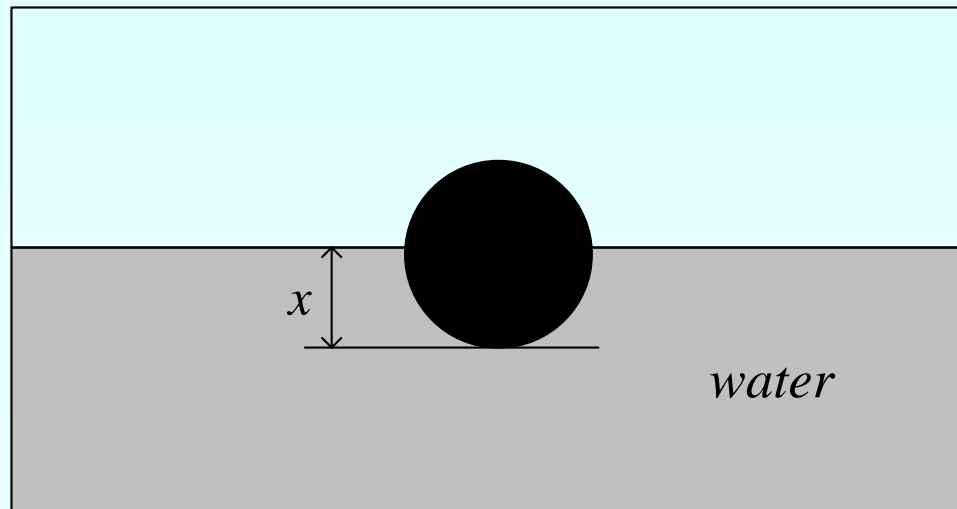
From the physics of the problem

$$0 \leq x \leq 2R$$

$$0 \leq x \leq 2(0.055)$$

$$0 \leq x \leq 0.11$$

Figure 2 :
Floating ball
problem





Let us assume

$$x_L = 0, x_U = 0.11$$

$$f(x_L) = f(0) = (0)^3 - 0.165(0)^2 + 3.993 \times 10^{-4} = 3.993 \times 10^{-4}$$

$$f(x_U) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993 \times 10^{-4} = -2.662 \times 10^{-4}$$

Hence,

$$f(x_L)f(x_U) = f(0)f(0.11) = (3.993 \times 10^{-4})(-2.662 \times 10^{-4}) < 0$$



Iteration 1

$$\begin{aligned}x_m &= \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \\&= \frac{0.11 \times 3.993 \times 10^{-4} - 0 \times (-2.662 \times 10^{-4})}{3.993 \times 10^{-4} - (-2.662 \times 10^{-4})} \\&= 0.0660\end{aligned}$$

$$\begin{aligned}f(x_m) &= f(0.0660) = (0.0660)^3 - 0.165(0.0660)^2 + (3.993 \times 10^{-4}) \\&= -3.1944 \times 10^{-5}\end{aligned}$$

$$f(x_L)f(x_m) = f(0)f(0.0660) = (+)(-) < 0$$

$$x_L = 0, x_U = 0.0660$$



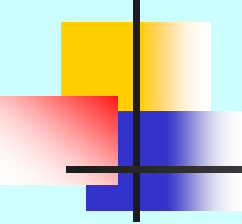
Iteration 2

$$\begin{aligned}x_m &= \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \\&= \frac{0.0660 \times 3.993 \times 10^{-4} - 0 \times (-3.1944 \times 10^{-5})}{3.993 \times 10^{-4} - (-3.1944 \times 10^{-5})} \\&= 0.0611\end{aligned}$$

$$\begin{aligned}f(x_m) &= f(0.0611) = (0.0611)^3 - 0.165(0.0611)^2 + (3.993 \times 10^{-4}) \\&= 1.1320 \times 10^{-5}\end{aligned}$$

$$f(x_L)f(x_m) = f(0)f(0.0611) = (+)(+) > 0$$

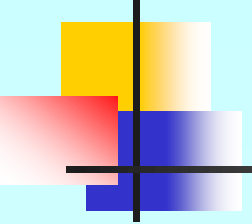
Hence, $x_L = 0.0611$, $x_U = 0.0660$



$$\epsilon_a = \left| \frac{0.0611 - 0.0660}{0.0611} \right| \times 100 \cong 8\%$$

Iteration 3

$$\begin{aligned} x_m &= \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \\ &= \frac{0.0660 \times 1.132 \times 10^{-5} - 0.0611 \times (-3.1944 \times 10^{-5})}{1.132 \times 10^{-5} - (-3.1944 \times 10^{-5})} \\ &= 0.0624 \end{aligned}$$



$$f(x_m) = -1.1313 \times 10^{-7}$$

$$f(x_L)f(x_m) = f(0.0611)f(0.0624) = (+)(-) < 0$$

Hence,

$$x_L = 0.0611, x_U = 0.0624$$

$$\epsilon_a = \left| \frac{0.0624 - 0.0611}{0.0624} \right| \times 100 \cong 2.05\%$$

Table 1: Root of $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$ for False-Position Method.

Iteration	x_L	x_U	x_m	$ \epsilon_a \%$	$f(x_m)$
1	0.0000	0.1100	0.0660	N/A	-3.1944×10^{-5}
2	0.0000	0.0660	0.0611	8.00	1.1320×10^{-5}
3	0.0611	0.0660	0.0624	2.05	-1.1313×10^{-7}
4	0.0611	0.0624	0.0632377619	0.02	-3.3471×10^{-10}

$$|\epsilon_a| \leq 0.5 \times 10^{2-m}$$

$$0.02 \leq 0.5 \times 10^{2-m}$$

$$0.04 \leq 10^{2-m}$$

$$\log(0.04) \leq 2 - m$$

$$m \leq 2 - \log(0.04)$$

$$m \leq 2 - (-1.3979)$$

$$m \leq 3.3979$$

$$\text{So, } m = 3$$

The number of significant digits at least correct in the estimated root of 0.062377619 at the end of 4th iteration is 3.



References

1. S.C. Chapra, R.P. Canale, Numerical Methods for Engineers, Fourth Edition, Mc-Graw Hill.

THE END